Physik IV - Lösungen - Serie 9

4. Mai 2011

① $\left\{-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega^2 x^2\right\}\psi(x) = E\psi(x)$

(a) $\psi_0(x) = C e^{-\alpha x^2/2}$

$$\frac{\partial^2}{\partial x^2}\psi_0(x) = C\frac{\partial}{\partial x}\left(-\alpha x\, e^{-\alpha x^2/2}\right)$$

$$= C\left(\alpha^2 x^2 - \alpha\right) e^{-\alpha x^2/2}$$

$$-\frac{\hbar^2}{2m}\left(\alpha^2 x^2 - \alpha\right) + \frac{1}{2}m\omega^2 x^2 = E_0$$

Powers of x must be equal for statement to be true for all x :—

$$\frac{\hbar^2}{2m}\alpha^2 = \frac{1}{2}m\omega^2 \Rightarrow \alpha = m\omega/\hbar \;///$$

$$E_0 = \frac{\hbar^2}{2m}\alpha = \frac{\cancel{\hbar^2}}{\cancel{2m}}\frac{\cancel{m}\omega}{\cancel{\hbar}} = \frac{1}{2}\hbar\omega \;///$$

$$C^2\int_{-\infty}^{\infty}\left(e^{-\alpha x^2/2}\right)^2 dx = C^2 \underbrace{\int_{-\infty}^{\infty} e^{-\alpha x^2} dx}_{\text{standard integral } = \left(\frac{\pi}{\alpha}\right)^{1/2}} = \underbrace{1}_{\text{normalisation of probabilities.}}$$

$$\Rightarrow C^2 = \left(\frac{\alpha}{\pi}\right)^{1/2}$$

$$\Rightarrow C = \left(\frac{\alpha}{\pi}\right)^{1/4}$$

$$\Rightarrow \psi_0(x) = \left(\frac{m\omega}{\hbar\pi}\right)^{1/4} e^{-m\omega x^2/2\hbar} \;|||.$$

(b) $\Delta x = \left(\langle x^2 \rangle - \langle x \rangle^2 \right)^{1/2}$

$\Delta p = \left(\langle p^2 \rangle - \langle p \rangle^2 \right)^{1/2}$

$$\int_{-\infty}^{\infty} e^{-\alpha x^2} dx = (\pi/\alpha)^{1/2}$$

$$\int_{-\infty}^{\infty} x^2 e^{-\alpha x^2} dx = \frac{1}{2} \sqrt{\pi} / \alpha^{3/2}$$

$$\int_{-\infty}^{\infty} x e^{-\alpha x^2} dx = 0 \quad \text{odd symmetry.}$$

$\langle x \rangle = 0$ (symmetry)

$$\langle x^2 \rangle = \left(\frac{\alpha}{\pi}\right)^{1/2} \int_{-\infty}^{\infty} e^{-\alpha x^2/2} x^2 e^{-\alpha x^2/2} dx$$

$$= \left(\frac{\alpha}{\pi}\right)^{1/2} \times \frac{1}{2\alpha} \left(\frac{\pi}{\alpha}\right)^{1/2} = \frac{1}{2\alpha} \; //$$

$$\Rightarrow \Delta x = \frac{1}{\sqrt{2\alpha}} \; //$$

$$\langle p^2 \rangle = \left(\frac{\alpha}{\pi}\right)^{1/2} \int_{-\infty}^{\infty} e^{-\alpha x^2/2} \left(-i\hbar \frac{\partial}{\partial x}\right)^2 e^{-\alpha x^2/2} dx$$

$$= \left(\frac{\alpha}{\pi}\right)^{1/2} (-\hbar^2) \int_{-\infty}^{\infty} (-\alpha + \alpha^2 x^2) e^{-\alpha x^2} dx$$

$$= \left(\frac{\alpha}{\pi}\right)^{1/2} (-\hbar^2) \left\{ -\alpha \underbrace{\int_{-\infty}^{\infty} e^{-\alpha x^2} dx}_{\sqrt{\pi/\alpha}} + \alpha^2 \underbrace{\int_{-\infty}^{\infty} x^2 e^{-\alpha x^2} dx}_{\frac{1}{2\alpha}\sqrt{\frac{\pi}{\alpha}}} \right\}$$

$$\Rightarrow \langle p^2\rangle = \left(\frac{\alpha}{\pi}\right)^{1/2}(-\hbar^2)\left\{-\alpha\sqrt{\frac{\pi}{\alpha}} + \alpha^2\frac{1}{2\alpha}\sqrt{\frac{\pi}{\alpha}}\right\}$$

$$= \left(\frac{\alpha}{\pi}\right)^{1/2}(-\hbar^2)\left(-\sqrt{\pi\alpha} + \frac{\sqrt{\pi\alpha}}{2}\right)$$

$$= \alpha\hbar^2/2 \quad //.$$

$$\langle p\rangle = \left(\frac{\alpha}{\pi}\right)^{1/2}\int e^{-\alpha x^2/2}\left(-i\hbar\frac{\partial}{\partial x}\right)e^{-\alpha x^2/2}\,dx$$

$$= \left(\frac{\alpha}{\pi}\right)^{1/2}(-i\hbar)\underbrace{\int_{-\infty}^{\infty}(-\alpha x)e^{-\alpha x^2}\,dx}_{\text{odd integral } = 0}$$

$$= 0$$

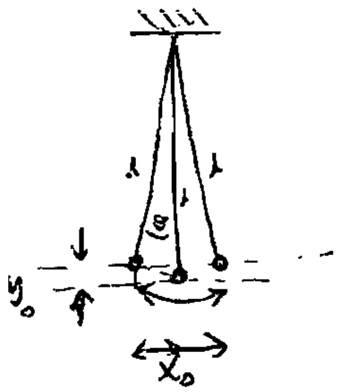
$$\Rightarrow \Delta p = \hbar\sqrt{\frac{\alpha}{2}} \quad //.$$

$$\Rightarrow \Delta x\,\Delta p = \hbar\sqrt{\frac{\alpha}{2}} \times \frac{1}{\sqrt{2\alpha}} = \frac{\hbar}{2} \quad //.$$

$\Rightarrow$ $\psi_0(x)$ is @ the Heisenberg uncertainty limit.

(c) Harmonic oscillators $E_n = \hbar\omega\left(n+\frac{1}{2}\right)$

Zero point energy $E_0 = \frac{\hbar\omega}{2} = \frac{\hbar}{2}\sqrt{\frac{g}{l}} = 3\times10^{-34}\,\text{J}.$



$mgy_0 = E_0 \Rightarrow y_0 = 3\times10^{-32}\,\text{m}.$

$\cos\theta = \frac{l-y_0}{l} \approx 1-\frac{\theta^2}{2} \Rightarrow \theta \approx \sqrt{\frac{2y_0}{l}}$

$x_0 = 2l\sin\theta \approx 2l\theta$

$\approx 2\sqrt{2ly_0} = 2.5\times10^{-16}\,\text{m}$ //.

A proton is about 1.5×10^{-15} m diameter, and an atom about 1×10^{-10} m....

(d) $y = 1$ mm

$E_n = \hbar\omega\left(n+\frac{1}{2}\right) = mgy$

$\Rightarrow n = \frac{mgy}{\hbar\omega} - \frac{1}{2} = 1.6\times10^{28}$!!

2) a) $\hat{a} = \sqrt{\frac{1}{2m\hbar\omega}}\,(m\omega\hat{x} + i\hat{p})$; $a^{+} = \sqrt{\frac{1}{2m\hbar\omega}}\,(m\omega\hat{x} - i\hat{p})$

$$[a, a^{+}] = \frac{1}{2m\hbar\omega}\,[m\omega\hat{x} + i\hat{p},\, m\omega\hat{x} - i\hat{p}] =$$

$$= \frac{1}{2m\hbar\omega}\Big\{\underbrace{[m\omega\hat{x}, m\omega\hat{x}]}_{0} + \underbrace{[\hat{p},\hat{p}]}_{0}$$

$$+ [m\omega\hat{x}, -i\hat{p}] + [i\hat{p}, m\omega\hat{x}]\Big\} =$$

$$= \frac{1}{2m\hbar\omega}(-2i)\,m\omega\,\underbrace{[\hat{x},\hat{p}]}_{i\hbar} = \underline{\underline{1}}$$

b) $$\psi_0(x) = \left(\frac{m\omega}{\hbar\pi}\right)^{\frac{1}{4}} e^{-m\omega x^2/2\hbar}$$

1. angeregter Zustand:

$$\psi_1(x) = a^{+}\psi_0(x)$$

$$= \sqrt{\frac{1}{2m\hbar\omega}}\,(m\omega\hat{x} - i\hat{p})\,\psi_0(x) \qquad \Big/\ \hat{p} \to -i\hbar\frac{\partial}{\partial x},\ \hat{x} \to x$$

$$= \sqrt{\frac{1}{2m\hbar\omega}}\left(m\omega x - \hbar\frac{\partial}{\partial x}\right)\psi_0(x)$$

$$= \left(\frac{\cancel{m\omega}}{\hbar\pi\, 4m^{\cancel{2}}\hbar^2\omega^{\cancel{2}}}\right)^{\frac{1}{4}}\left[m\omega x\, e^{-\frac{m\omega x^2}{2\hbar}} + \cancel{\hbar}\,\frac{m\omega x}{\cancel{\hbar}}\, e^{-\frac{m\omega x^2}{2\hbar}}\right]$$

$$= \left(4\frac{m^3\omega^3}{\hbar^3\pi}\right)^{\frac{1}{4}} x\, e^{-\frac{m\omega x^2}{2\hbar}}$$

c) $\langle \hat{O} \rangle = \langle \psi_1 | \hbar\omega (a^\dagger a + \frac{1}{2}) | \psi_1 \rangle = \frac{\hbar\omega}{2} + \langle \psi_1 | \hbar\omega a^\dagger a | \psi_1 \rangle$

$$= \int_{-\infty}^{\infty} \psi_1^*(x) \left(\hbar\omega a^\dagger a\right) \psi_1(x)\, dx + \frac{\hbar\omega}{2}$$

mit $a^\dagger a \, \psi_n(x) = n \cdot \psi_n(x)$ folgt

$$\langle \hat{O} \rangle = \hbar\omega \underbrace{\int_{-\infty}^{\infty} \psi_1^*(x)\, \psi_1(x)\, dx}_{1} + \frac{\hbar\omega}{2} = \frac{3\hbar\omega}{2}$$

d) $\psi(x,t) = c_0 \, \psi_0(x,t) + c_1 \, \psi_1(x,t) =$

$$= c_0 \, e^{-\frac{i}{\hbar} E_0 t} \psi_0(x) + c_1 \, e^{-\frac{i}{\hbar} E_1 t} \psi_1(x) =$$

$$= c_0 \, e^{-i \frac{\omega t}{2}} \psi_0(x) + c_1 \, e^{-i \frac{3\omega t}{2}} \psi_1(x)$$

$\psi(x,t) = \psi(x,0)$ für $\omega t = 2\pi$ (langsamere Oszillation)

$$\Rightarrow t \equiv \tau = \frac{4\pi}{\omega} \quad \left(\frac{3\omega\tau}{2} = \frac{3\omega 4\pi}{2\omega} = 6\pi \right)$$

$$\tau = \frac{2\pi}{2\pi 10^{10}} = 2 \cdot 10^{-10} \mathrel{\hat{=}} \underline{0.\ \text{ns}}$$

3a) $\langle \hat{L}^2 \rangle = \int d^3r \; \psi^*_{nlm}(r,\theta,\varphi) \; \hat{L}^2 \; \psi_{nlm}(r,\theta,\varphi) =$

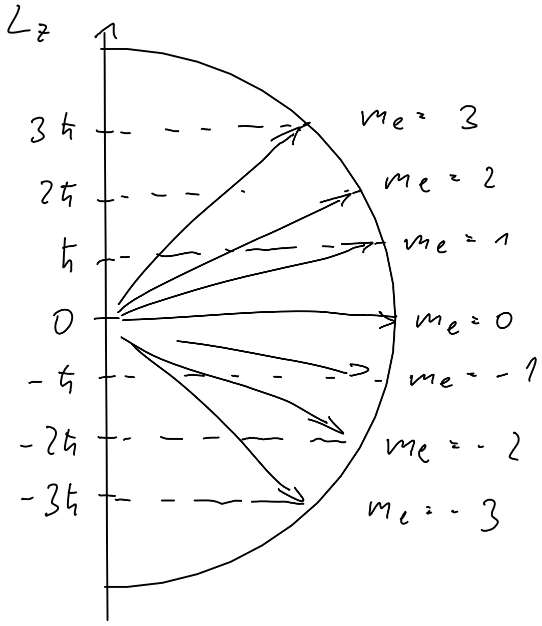
$|l=3| \quad = \int d^3r \; |R_{n3}(r)|^2 \; Y_{3m}(\theta,\varphi) \; \hat{L}^2 \; Y_{3m}(\theta,\varphi) =$

$= \int d^3r \; |R_{n3}(r)|^2 \; Y_{3m}(\theta,\varphi) \left[\hbar^2 l(l+1)\right] Y_{3m}(\theta,\varphi) =$

$= \hbar^2 l(l+1) \underbrace{\int d^3r \; \psi^*_{n3m}(r,\theta,\varphi) \; \psi_{n3m}(r,\theta,\varphi)}_{1}$

$= \hbar^2 l(l+1)$

$|\vec{L}| = \sqrt{\langle L^2 \rangle} = \hbar \sqrt{l(l+1)} \Rightarrow \sqrt{12}\,\hbar$



b) $\hat{L}_z \; \phi_{ml} = \hbar m_l \; \phi_{ml}$

aus Serie 7, Aufgabe 2

apply the equation to ϕ_{ml}:

$\hat{L}_z \hat{L}_\pm \phi_{ml} = \hat{L}_\pm \underbrace{\hat{L}_z \phi_{ml}}_{\hbar m_l \phi_{ml}} \pm \hbar \hat{L}_\pm \phi_{ml}$

$\Rightarrow \hat{L}_z \left(\hat{L}_\pm \phi_{ml}\right) = \hbar (m_l \pm 1) \left(\hat{L}_\pm \phi_{ml}\right)$

Solution 3(c) Why $L_+ \Phi_{ll} = 0$?

From (8) $L_+ \Psi_{ml} = \Psi_{(m+1)l}$

$L_+ \Phi_{ll} = 0$ because there is no states with $m > l$ $(m: -l, -(l-1), \ldots 0, \ldots l)$

First apply the condition

$L_z \Phi_{ll} = \hbar l \Phi_{ll} \Rightarrow -i\hbar \frac{\partial}{\partial \varphi} \Phi_{ll} = \hbar l \Phi_{ll}$

$\Rightarrow \Phi_{ll} = \text{const}\, e^{il\phi} = A e^{il\phi}$

constant A doesnot depend on ϕ but may depend on θ

Apply second condition:

$L_+ \Phi_{ll} = \left(\frac{\partial}{\partial \theta} + i \cot\theta \frac{\partial}{\partial \phi} \right) A(\theta) e^{il\phi} = 0 \Rightarrow$

$e^{il\phi} \left(\frac{\partial}{\partial \theta} A(\theta) + i \cot\theta A(\theta) (il) \right) = 0 \Rightarrow$

$\frac{\partial}{\partial \theta} A(\theta) = l \cot\theta A(\theta) \Rightarrow$

$\frac{dA(\theta)}{A(\theta)} = l \cot\theta \, d\theta \Rightarrow \int \frac{dA}{A} = l \int \cot\theta \, d\theta$

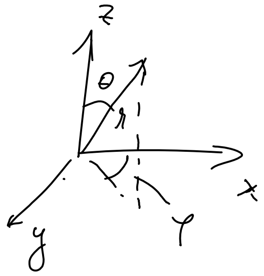
$\Rightarrow \ln A = l \ln(\sin\theta) + \text{const} \Rightarrow$

$A(\theta) = (\sin\theta)^l \cdot B$ where B is a constant

$\Rightarrow \Phi_{ll} = A(\theta) e^{il\phi} = B (\sin\theta)^l e^{il\phi}$

B can be found from normalization

4) probability density $= |\psi_{nlm}(r,\theta,\varphi)|^2$



$$\psi_{nlm} = R_{n,l}(r)\, Y_{l,m}(\theta,\varphi)$$

$\uparrow$ radial part depends only on r

$\Rightarrow$ symmetric about z $\forall\, n, l$

$$Y_{00}(\theta,\varphi) = \sqrt{\frac{1}{4\pi}}$$

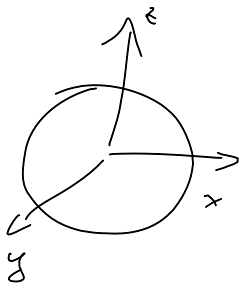
$$Y_{10}(\theta,\varphi) = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_{1\pm 1}(\theta,\varphi) = \pm\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{\pm i\varphi}$$

$$|\psi_{nlm}(r,\theta,\varphi)|^2 = R_{nl}^2(r)\ \underbrace{Y_{l,m}^{*}\, Y_{lm}}_{\text{interesting part}}$$

$Y_{00}^{*} Y_{00} = \frac{1}{4\pi}$ ... spherical symmetric

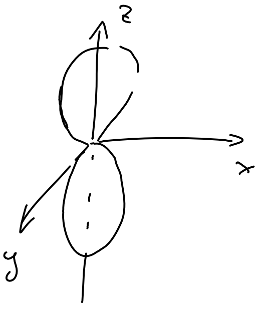
$\Rightarrow$ also symmetric about z-axis



$Y_{10}^{*} Y_{10} = \frac{3}{4\pi}\cos^2\theta$

no φ-dependence

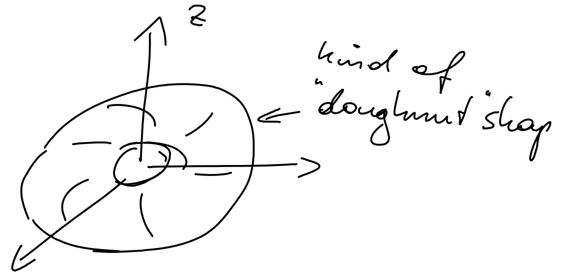
$\Rightarrow$ symmetric about z-axis



$$Y_{1\pm1}^* Y_{1\pm1} = \frac{3}{8\pi} \sin^2\Theta \; e^{\pm i\varphi} e^{\mp i\varphi} = \frac{3}{8\pi} \sin^2\Theta$$

=> φ dependence drops out by calculating the modulus squared

=> also symmetric about z-axis



Time-dependent Harmonic Oscillator

Time-independent harmonic oscillator

In[100]:=
```
division = 100.;
mass = 1;
kconst = 30;
bound = 2;
```

In[113]:=
```
xvalues = Table[x, {x, -bound, bound, 2 bound / division}];
```

In[104]:=
```
DiffMatrix[m_, dx_, Vx_] :=
  (DiagonalMatrix[-ℏ^2/(2 m) (-2)/dx^2 + Vx] + DiagonalMatrix[Table[-ℏ^2/(2 m) 1/dx^2, {Length[Vx] - 1}], 1] +
     DiagonalMatrix[Table[-ℏ^2/(2 m) 1/dx^2, {Length[Vx] - 1}], -1]) /. ℏ → 1
```

In[105]:=
```
Vfunc[k_, x_, amp_, ωp_, ϕ_, t_] := (k (1 + amp Cos[ωp t + ϕ]))/2 x^2
(*Frequency:*)
ω = Sqrt[D[Vfunc[kconst, x, 0, 0, 0, 0], {x, 2}] / mass];
tfinal = 2 π / ω;
```

In[108]:=
```
Varray[k_, amp_, ωp_, ϕ_, t_, boundary_, dx_] :=
  Table[Vfunc[k, x, amp, ωp, ϕ, t], {x, -boundary, boundary, dx}];
```

Set up the Matrix for the Problem and calculate the Eigenfunctions and Eigenvectors:

In[109]:=
```
MProblem[amp_, ωp_, ϕ_, t_] := DiffMatrix[mass,
   2 bound / division, Varray[kconst, amp, ωp, ϕ, t, bound, 2 bound / division]];
```

Berechne den Grundzustand für t = 0, amp=0:

In[110]:=
```
tzero = Eigensystem[MProblem[0, 0, 0, 0]];
{evalstzero, evectstzero} = Sort[Transpose[tzero], #1[[1]] < #2[[1]] &] // Transpose;
stateenergies = Take[evalstzero, 5]
```

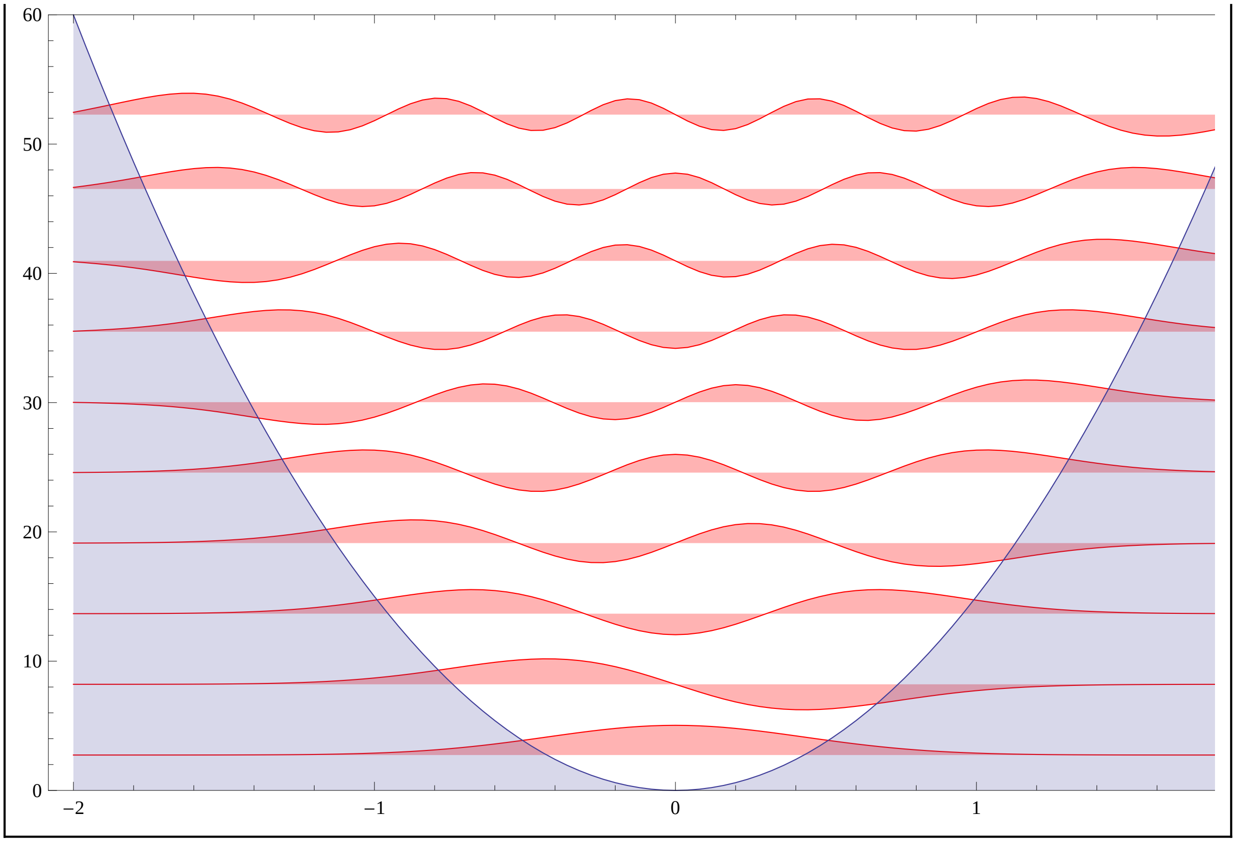
Out[112]=
```
{2.73711, 8.20833, 13.6735, 19.1328, 24.5862}
```

Plot the Eigenvectors on top of the Eigenenergies :

In[114]:=

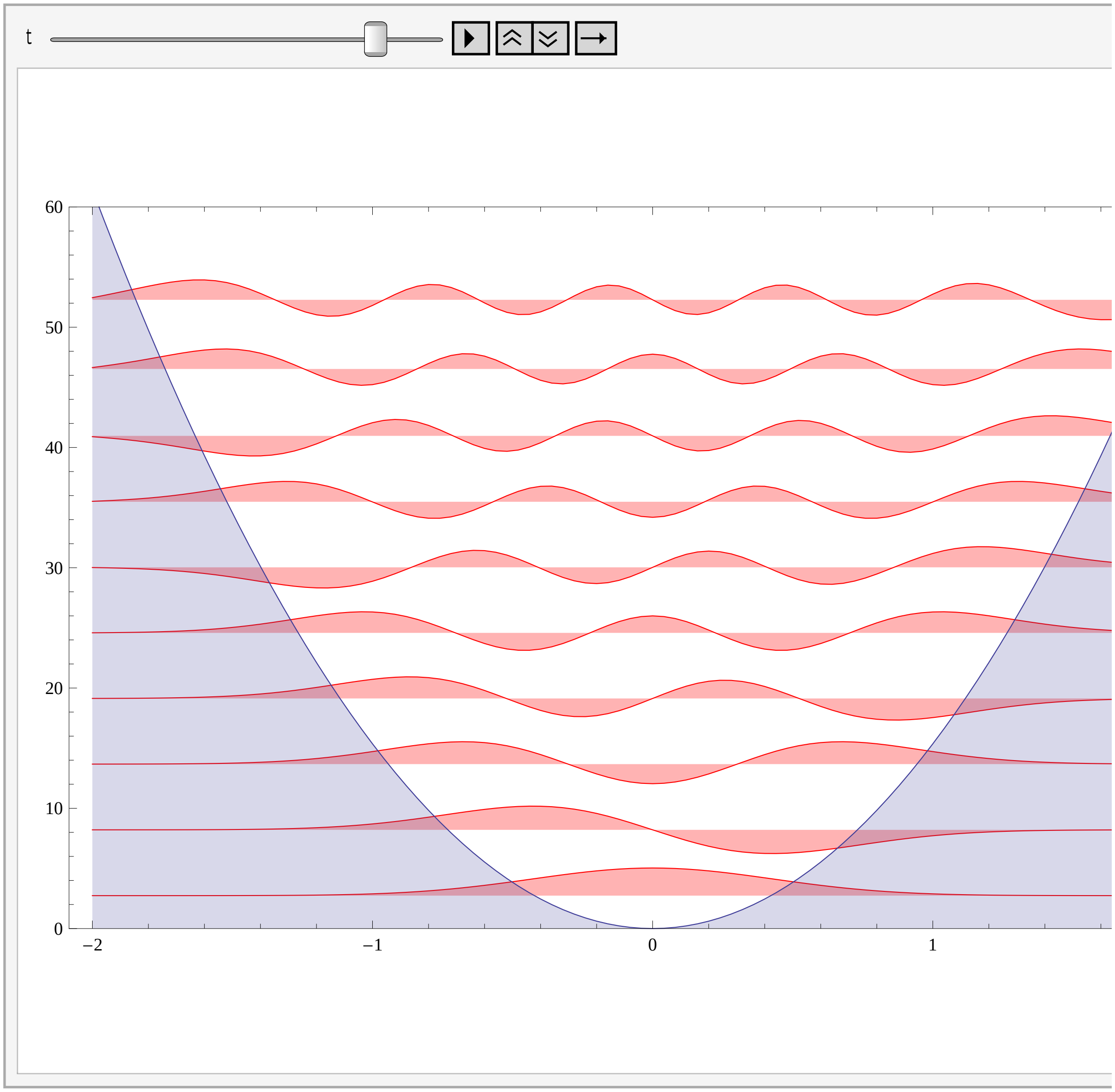
```
potplot[k_, amp_, ωp_, ϕ_, t_] :=
  ListPlot[{xvalues, Varray[k, amp, ωp, ϕ, t, bound, 2 bound / division]}ᵀ,
   Filling → 1.1 Min[Varray[k, amp, ωp, ϕ, t, bound, 2 bound / division]],
   Joined → True, Frame → True, PlotRange → {0, 60}, GridLines → {None, stateenergies},
   GridLinesStyle → Directive[Thickness[0.002], Red], Axes → False];
ScaleFactor = 10;
stateplot =
  ListPlot[Table[{xvalues, ScaleFactor evectstzero[[i]] + evalstzero[[i]]}ᵀ, {i, 1, 10}],
   Joined → True, Filling → Table[n → evalstzero[[n]], {n, 1, 10}],
   PlotStyle → Red, FillingStyle → Directive[Red, Opacity[0.3]],
   PlotRange → All, ImageSize → 600, Frame → True, Axes → False];
Show[stateplot, potplot[kconst, 0, 0, 0, 0], PlotRange → {Automatic, {0, 60}}]
```

Out[117]=



In[118]:=
```
Animate[Show[stateplot, potplot[kconst, 0.2, 2 ω, 0, t],
  PlotRange → {Automatic, {0, 60}}], {t, 0, 10}]
```

Out[118]=



Time-dependent harmonic oscillator :

parameters :

In[119]:=
```
ampconst = 0.2;
ϕconst = 0;
```

Start in the groundstate :

In[121]:=
```
uinitial = evectstzero[[1]];
```

parametric driving at twice the harmonic oscillator frequency

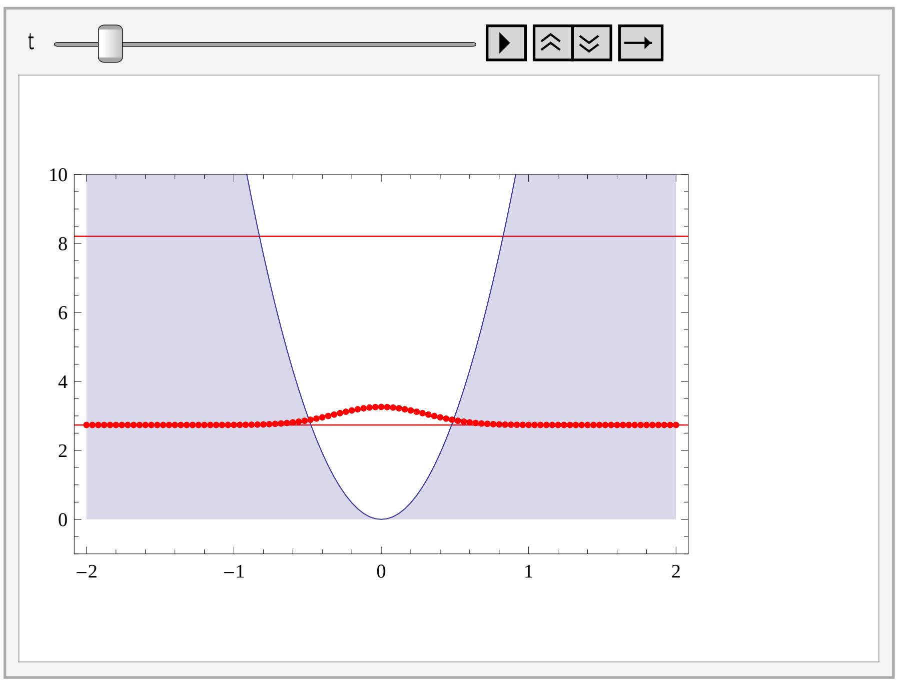
In[124]:=
```
answertdepground[t_] =
 u[t] /. First @@ NDSolve[{-ⅈ Evaluate[MProblem[ampconst, 2 ω, ϕconst, t]].u[t] == u'[t],
     u[0] == uinitial}, u[t], {t, 0, 3 tfinal}]
```

Out[124]=
```
InterpolatingFunction[{{0., 3.44144}}, <>][t]
```

Aufenthalswahrscheinlichkeit der Wellenfunktion :

In[127]:=
```
Animate[Show[potplot[kconst, ampconst, 2 ω, ϕconst, t],
   ListPlot[{xvalues, evalstzero[[1]] + ScaleFactor Abs[answertdepground[t]]^2}ᵀ,
    PlotStyle → Red, FillingStyle → Directive[Red, Opacity[0.3]]], PlotRange → {-1, 10}],
 {t, 0, 3 tfinal}, AnimationRunning → False, AnimationRate → 0.1]
```

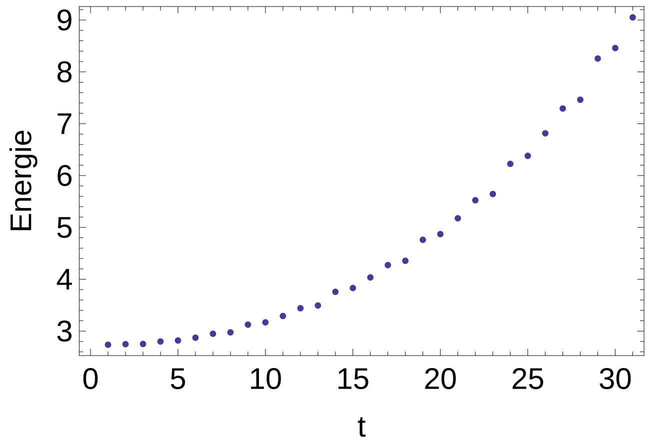
Out[127]=



Energie :

In[128]:=
```
ListPlot[Table[answertdepground[t]*. MProblem[0, 0, 0, 0].answertdepground[t],
  {t, 0, 3 tfinal, 0.1 tfinal}], BaseStyle → {FontFamily → "Arial", FontSize → 14},
 Frame → True, FrameLabel → {"t", "Energie"}]
```

Out[128]=



parametric driving at the harmonic oscillator frequency

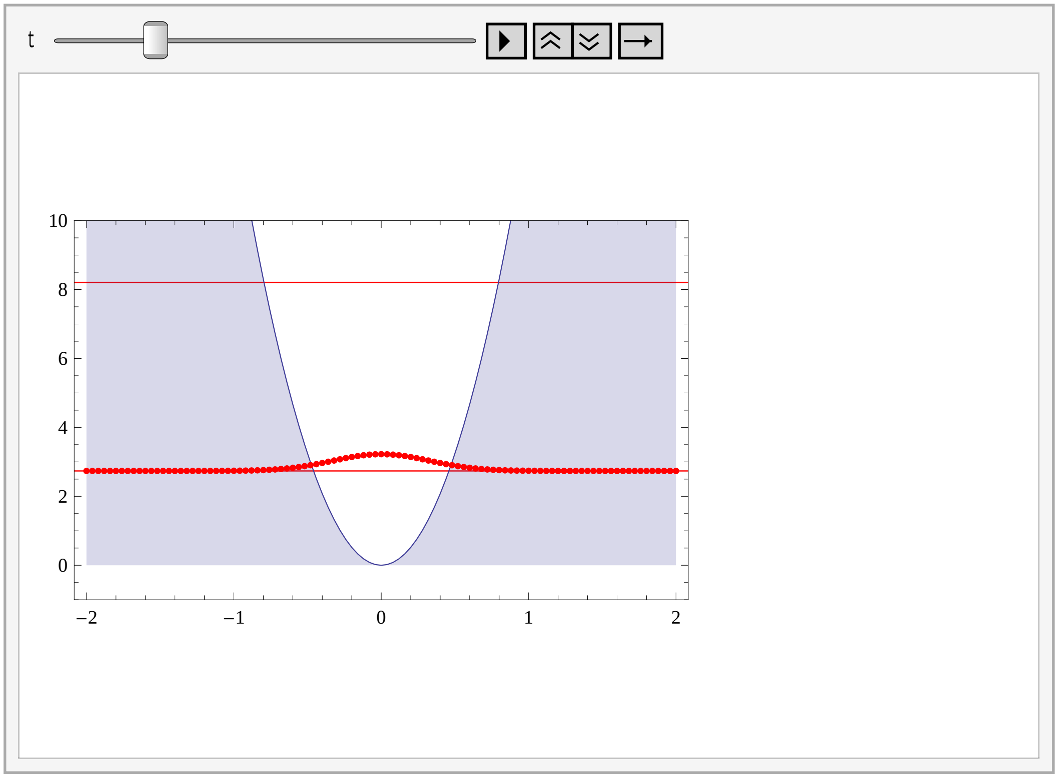
In[129]:=
```
answertdepgrounddeg[t_] =
 u[t] /. First @@ NDSolve[{-ⅈ Evaluate[MProblem[ampconst, ω, ϕconst, t]].u[t] == u'[t],
     u[0] == uinitial}, u[t], {t, 0, 3 tfinal}]
```

Out[129]=
```
InterpolatingFunction[{{0., 3.44144}}, <>][t]
```

Aufenthalswahrscheinlichkeit der Wellenfunktion :

In[131]:=
```
Animate[Show[potplot[kconst, ampconst, ω, ϕconst, t],
  ListPlot[{xvalues, evalstzero[[1]] + ScaleFactor Abs[answertdepgrounddeg[t]]^2}^T,
   PlotStyle → Red, FillingStyle → Directive[Red, Opacity[0.3]]],
  PlotRange → {-1, 10}], {t, 0, 3 tfinal}, AnimationRunning → False]
```

Out[131]=



Energie :

In[132]:=

```
ListPlot[Table[answertdepgrounddeg[t]*. MProblem[0, 0, 0, 0].answertdepgrounddeg[t],
   {t, 0, 3 tfinal, 0.1 tfinal}],
  BaseStyle → {FontFamily → "Arial", FontSize → 14},
  Frame → True, FrameLabel → {"t", "Energie"}]
```

Out[132]=