1.6. Multi Qubit States

Mittwoch, 22. Februar 2012
17:52

How many different states can two classical or quantum mechanical states be in?

register of $n=2$ classical bits

Bit A	Bit B
0	0
0	1
1	0
1	1

} 2^n different states

note: only one state is realized at any given time

register of $n=2$ quantum bits

Qubit A	Qubit B
$\|0\rangle$	$\|0\rangle$
$\|0\rangle$	$\|1\rangle$
$\|1\rangle$	$\|0\rangle$
$\|1\rangle$	$\|1\rangle$

} 2^n basis states

BUT: quantum register can be in any superposition of basis states

Tensor product: (formal description of quantum system composed of $n=2$ subsystems)

$$|\psi\rangle = |A\rangle \otimes |B\rangle = |AB\rangle$$

e.g.: $|A\rangle = \alpha_A|0\rangle + \beta_A|1\rangle$; $|B\rangle = \alpha_B|0\rangle + \beta_B|1\rangle$

$$|\psi\rangle = \alpha_A\alpha_B|00\rangle + \alpha_A\beta_B|01\rangle + \beta_A\alpha_B|10\rangle + \beta_A\beta_B|11\rangle$$

with $\sum_{ij}|\alpha_{ij}|^2 = 1$ (normalization - condition for probabilities)

Information content of many Qubit state:

register of n qubits: How many classical bits are needed to describe such a system?

- 2^n basis states
- general superposition described by $(2^n - 1)$ complex coefficients
- assuming $2 * 32$ bit for each coefficient

for $n = 300$: $2 \cdot 2^{32} \cdot 2^{300} \sim 10^{10+90}$ bits

-> larger than number of atoms in the universe

-> impossible to store information about state classically

-> difficult to simulate QM on a classical computer

-> use a quantum computer!

Entangled Qubit States:

Definition: An entangled state of a composite system is a state that cannot be written as a tensor product state of the component systems

product state: $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$ with $|\psi_1\rangle = \alpha_1 |0\rangle + \beta_1 |1\rangle$
$|\psi_2\rangle = \alpha_2 |0\rangle + \beta_2 |1\rangle$

$$= \alpha_1\alpha_2 |00\rangle + \alpha_1\beta_2 |01\rangle + \beta_1\alpha_2 |10\rangle + \beta_1\beta_2 |11\rangle$$

entangled state: $|\psi\rangle = \frac{1}{\sqrt{2}} \left(|00\rangle + |11\rangle \right)$

Can this state be written in the form above?

$$\left. \begin{array}{l} \alpha_1\alpha_2 = \beta_1\beta_2 = \frac{1}{\sqrt{2}} \\ \Rightarrow \alpha_1\beta_2 \neq 0 \ \wedge \ \beta_1\alpha_2 \neq 0 \ ! \end{array} \right\} \Rightarrow \text{not a product state!}$$

*) How are such states created? How would you go about creating such states?

*) What are their properties?

Correlations of Entangled States

Measurement of individual qubits in an entangled state

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left(|01\rangle + |10\rangle \right)$$

Which outcomes are possible, if ground state ($P_0^{(1)} = |0\rangle\langle 0|$) is measured on first qubit?

'0' with probability

$$p^{(1)}(0) = \langle \psi | P_0^{(1)} | \psi \rangle = \langle \psi | P_0^{(1)} \otimes \mathbb{1} | \psi \rangle =$$

identity op for not measured qubits

$$= \frac{1}{2} \left(\langle 01| + \langle 10| \right) P_0^{(1)} \otimes \mathbb{1} \left(|01\rangle + |10\rangle \right) =$$

$$= \frac{1}{2} \left(\underbrace{\langle 0 | P_0^{(1)} | 0 \rangle}_{1} \underbrace{\langle 1 | \mathbb{1} | 1 \rangle}_{1} + \langle 0 | P_0^{(1)} | 1 \rangle \langle 1 | 1 \rangle + \ldots \right) = \frac{1}{2}$$

post measurement state:

$$|\psi'\rangle = \frac{P_0^{(1)} |\psi\rangle}{\sqrt{p^{(1)}(0)}} = \frac{\frac{1}{\sqrt{2}} |01\rangle}{\frac{1}{\sqrt{2}}} = |01\rangle$$

measure excited state of 2nd qubit:

$$\langle \psi' | P_1^{(2)} | \psi' \rangle = \langle \psi' | \mathbb{1} \otimes P_1^{(2)} | \psi' \rangle = \langle 0 | \mathbb{1} | 0 \rangle \langle 1 | P_1^{(2)} | 1 \rangle = 1$$

⇒ 100% certainty that 2nd qubit is in its excited state

⇒ same holds for outcome '1': if 1st qubit is found in $|1\rangle$, 2nd is with 100% certainty in $|0\rangle$!

⇒ quantum correlations which are stronger than classical ones!
(Bell inequalities)

Question: What is the value of $\langle \hat{\sigma}_z^{(1)} \rangle$ for $|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + e^{i\varphi}|\downarrow\uparrow\rangle)$?

$$\langle \psi | \sigma_z^{(1)} \otimes \mathbb{1} | \psi \rangle = \frac{1}{2}\Big\{ \underbrace{\langle \uparrow | \sigma_z^{(1)} | \uparrow \rangle}_{+1} \underbrace{\langle \downarrow | \downarrow \rangle}_{1} + \langle \uparrow | \sigma_z^{(1)} | \downarrow \rangle \underbrace{\langle \downarrow | \uparrow \rangle}_{0} e^{i\varphi} +$$

$$\langle \downarrow | \sigma_z^{(1)} | \uparrow \rangle \underbrace{\langle \uparrow | \downarrow \rangle}_{0} e^{-i\varphi} + \underbrace{\langle \downarrow | \sigma_z^{(1)} | \downarrow \rangle}_{-1} \underbrace{\langle \uparrow | \uparrow \rangle}_{1} \Big\} = 0$$

$\Rightarrow$ single particle carries no information about the spin!

$\Rightarrow$ relative phase φ is irrelevant!

State that describes single particle component:

$$\langle A^{(1)} \rangle = \langle \psi | A^{(1)} \otimes \mathbb{1} | \psi \rangle =$$

$$= \sum_{ij} c_{ij}^* \langle i | \otimes \langle j | \Big(A^{(1)} \otimes \sum_m |m\rangle\langle m| \Big) \sum_{kl} c_{kl} |k\rangle \otimes |l\rangle$$

$$= \sum_{ij,kl,m} c_{ij}^* c_{kl} \langle i | A^{(1)} | k \rangle \underbrace{\langle j | m \rangle}_{\delta_{jm}} \underbrace{\langle m | l \rangle}_{\delta_{ml}} =$$

$$= \sum_{i,km} c_{im}^* c_{km} \langle i | A^{(1)} | k \rangle = \mathrm{Tr}\,[\rho \hat{A}^{(1)}]$$

$$\text{with } \rho = \sum_{ik} \tilde{c}_{ik} |k\rangle\langle i| \qquad \tilde{c}_{ik} = \sum c_{im}^* c_{km}$$

$$\mathrm{Tr}\,[\rho \hat{A}^{(1)}] = \sum_n \langle n | \sum_{ik} \tilde{c}_{ik} |k\rangle\langle i| \hat{A}^{(1)} | n \rangle =$$

$$= \sum_{ik,n} \tilde{c}_{ik} \underbrace{\langle n | k \rangle}_{\delta_n} \langle i | \hat{A}^{(1)} | n \rangle =$$

$$= \sum_{ik} \tilde{c}_{ik} \langle i | \hat{A}^{(1)} | n \rangle \qquad \square$$

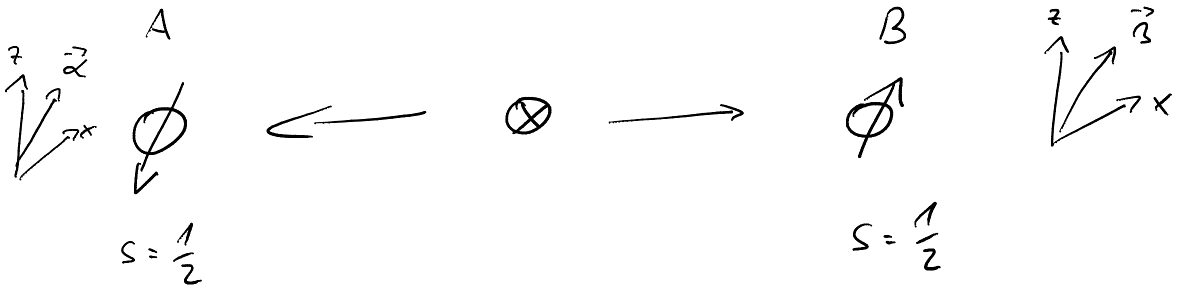
ρ – 'mixed' state due to lack of knowledge of the second particle

1.8. Bell Inequalities

Montag, 19. September 2011
12:24

singlett Zustand (rotational symmetric)

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_1 |\downarrow\rangle_2 - |\downarrow\rangle_1 |\uparrow\rangle_2\right)$$



Superposition of 'particle 1 having spin $\oplus \frac{\hbar}{2}$ - particle 2 $\ominus \frac{\hbar}{2}$'

and 'particle 2 $\oplus \frac{\hbar}{2}$ - particle 1 $\ominus \frac{\hbar}{2}$'

Measurement of particle 1 along direction $\vec{\alpha}$: $m = \pm \frac{\hbar}{2}$

— " — 2 — " — $m = \mp \frac{\hbar}{2}$

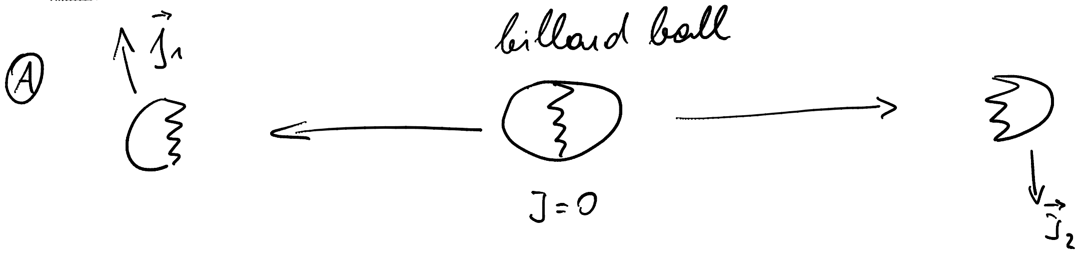
=> Spin is not determined by wavefunction ψ, only correlations

=> measurement 1 determines instantaneously measurement 2!

=> Is the information given by ψ incomplete?

=> Can we introduce hidden local variables (LHV) to describe state more completely?

Bell inequalities for classical system



Conservation of angular momentum: $\vec{J}_1 = -\vec{J}_2$

Measurement of ang. moment along axis $\vec{\alpha}, \vec{\beta}$:

$$a = \operatorname{sgn}(\vec{\alpha}\cdot\vec{J}_1) \qquad b = \operatorname{sgn}(\vec{\beta}\cdot\vec{J}_2)$$

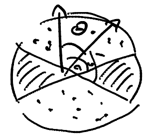
$\vec{\alpha} = \vec{\beta}$: $\quad -1 \quad\quad +1$

$\quad\quad\quad\quad +1 \quad\quad -1$

correlation: $\langle a\cdot b\rangle = \frac{1}{N}\sum_j^N a_j\cdot b_j = \frac{1}{N}\sum_j^N(\pm 1)(\mp 1) = \frac{-N}{N} = -1$

$\vec{\alpha} \neq \vec{\beta}$:

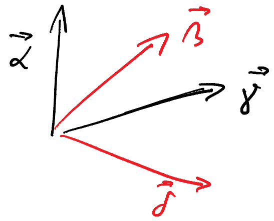
$$\langle a\cdot b\rangle = -1 + \frac{2\theta}{\pi}$$



$$\frac{\because}{////} = \frac{2(\pi-\theta)}{2\theta} = \frac{\pi-\theta}{\theta}$$

$$\langle a\cdot b\rangle = p_1\cdot 1 + p_{-1}\cdot(-1) = \frac{1}{\pi}(\theta-(\pi-\theta))$$

Ungleichung mit 2 verschiedenen Messwinkel pro Seite ($\vec{\alpha},\vec{\gamma}$ & $\vec{\beta},\vec{\delta}$):



$$a = \operatorname{sgn}(\alpha J_1) \qquad b = \operatorname{sgn}(\beta J_2)$$
$$c = \operatorname{sgn}(\gamma J_1) \qquad d = \operatorname{sgn}(\delta J_2)$$

Possible values of $S_j = a_j b_j + b_j c_j + c_j d_j - a_j d_j = ?$

Hint: $S_j = b_j\underbrace{(a_j + c_j)}_{?} + d_j\underbrace{(c_j - a_j)}_{?}$ für $a_j = \pm 1$ $c_j = \pm 1$

Answer: (a) 0 (b) 2 ✓ (c) 4

Assumptions: *) Values of $\vec{J}_1, \vec{J}_2$ are determined for each experiment

*) no influence of measurement 1 on meas. 2

Quantum model:

Measurement along $\vec{\alpha}, \vec{\beta}$:

$$\left[\ \vec{\alpha}\cdot\vec{\sigma} = \begin{pmatrix} \sin\alpha \\ 0 \\ \cos\alpha \end{pmatrix} \cdot \begin{pmatrix} \hat{\sigma}_x \\ \hat{\sigma}_y \\ \hat{\sigma}_z \end{pmatrix} = \sin\alpha\, \hat{\sigma}_x + \cos\alpha\, \hat{\sigma}_z \ \right]$$

$$\langle a \cdot b \rangle \overset{QM}{=} \langle \psi | \vec{\alpha}\cdot\vec{\sigma} \otimes \vec{\beta}\vec{\sigma} | \psi \rangle = -\cos(\alpha - \beta)$$

$$\Rightarrow S_{QM} = |\langle ab \rangle + \langle bc \rangle + \langle cd \rangle - \langle ad \rangle| =$$

$$= |\cos(\alpha - \beta) + \cos(\beta - \gamma) + \cos(\gamma - \delta) - \cos(\alpha - \delta)|$$

$$\left.\begin{aligned} \alpha &= 0 \\ \beta &= \tfrac{\pi}{4} \\ \gamma &= \tfrac{\pi}{2} \\ \delta &= \tfrac{3\pi}{4} \end{aligned}\right\} \quad |\underbrace{\cos\tfrac{\pi}{4}}_{\frac{1}{\sqrt{2}}} + \cos\tfrac{\pi}{4} + \cos\tfrac{\pi}{4} - \underbrace{\cos\tfrac{3\pi}{4}}_{-\frac{1}{\sqrt{2}}}| = \frac{4}{\sqrt{2}} = \underline{\underline{2\cdot\sqrt{2}}}$$

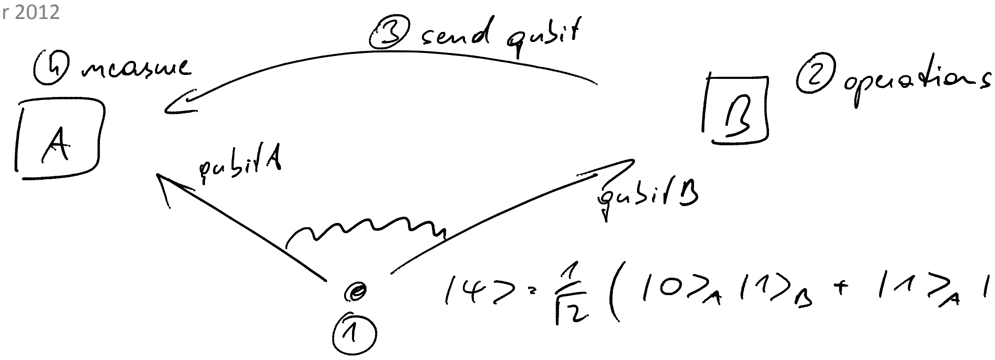
$\Rightarrow$ QM violates Bell Inequalities !!

(1) 'spooky-action-at a distance' (non locality)

(2) state is only determined at time of measurement (non realistic)

1. 9. Superdense coding - entanglement as ressource

Montag, 27. Februar 2012
15:11



$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle_A|1\rangle_B + |1\rangle_A|0\rangle_B\right)$$

1) share entangled pair of qubits

2) Bob performs one of 4 local operations on his qubit

a) do nothing: $\mathbb{1}|\psi\rangle = |\psi^+\rangle$

b) $\sigma_x^{(2)}$: $\sigma_x^{(2)}|\psi\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) \equiv |\phi^+\rangle$

c) $\sigma_z^{(2)}$: $\sigma_z^{(2)}|\psi\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right) \equiv |\psi^-\rangle$

d) $i\sigma_y^{(2)}$: $\sigma_x^{(2)}\sigma_z^{(2)}|\psi\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle - |11\rangle\right) \equiv |\phi^-\rangle$

3) Bobs sends qubit to Alice (<u>ONE</u> qubit)

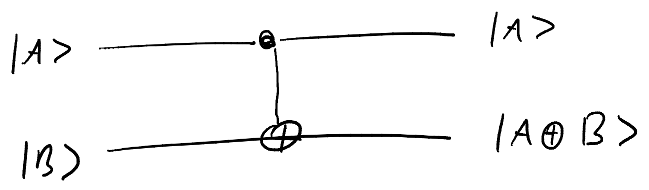
4) Alice measures state on <u>both</u> qubits and finds 4 outcomes

(proposed by Wiesner & Bennett)

1.10. CNOT gate: A universal 2-qubit gate

Dienstag, 06. März 2012
18:46

controlled NOT gate:



control qubit
target qubit

truth table:

$\|A\rangle \otimes \|B\rangle$	$\|A\rangle \otimes \|A \oplus B\rangle$
$\|00\rangle$	$\|00\rangle$
$\|01\rangle$	$\|01\rangle$
$\|10\rangle$	$\|11\rangle$
$\|11\rangle$	$\|10\rangle$

- reversible
- universal together with single qubit operations

$|AB\rangle$ $\qquad$ $|A \oplus B\rangle$

↑ addition modulo 2

Universality: Any multi qubit logic gate can be composed of CNOT and single qubit gates $(\hat{X}, \hat{Y}, \hat{Z})$

Creation of entangled state: