I.4. Single Qubit Gates

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Circuit representation:

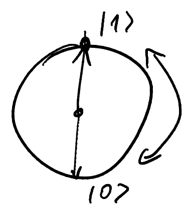
$$|\psi\rangle_{IN} \;—\; \boxed{\hat{U}} \;—\; |\psi\rangle_{OUT}$$

Work out how specific operations are represented on the Bloch sphere:

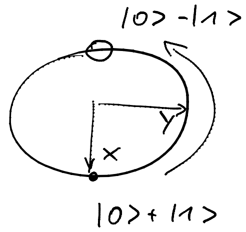
*) identity: $\hat{U} = \hat{I} = \mathbb{1}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

*) bit flip: $\hat{U} = \hat{X} = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\hat{=}$ NOT operation

(rotation by π about x-axis)



*) phase flip: $\hat{U} = \hat{Z} = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$



(top view)

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

*) conjugate bit flip: $\hat{U} = \hat{Y} = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$
(bit-phase flip)

$|0\rangle \rightarrow -i|1\rangle$

$|1\rangle \rightarrow i|0\rangle$

$\sigma_x, \sigma_y, \sigma_z$ ... Pauli matrices

$\{\mathbb{1}, \sigma_x, \sigma_y, \sigma_z\}$ basis set for operators on a single qubit

Hadamard gate: generation of superposition from $|0\rangle$ or $|1\rangle$

$$\hat{H} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}} (\hat{X} + \hat{Z})$$

$\hat{=}$ rotation by π about $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$-axis

$$|0\rangle \; — \boxed{\hat{H}} — \; \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$|1\rangle \; — \boxed{\hat{H}} — \; \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

$\Rightarrow$ used in many quantum algorithms to prepare (initial) superposition state

Single Qubit Dynamics

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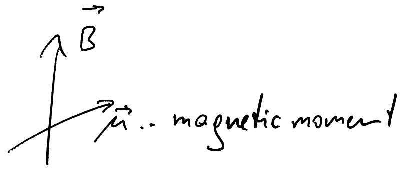
time-evolution given by Schrödinger equation

$$\boxed{\hat{H}|\psi\rangle = i\hbar \frac{\partial}{\partial t}|\psi\rangle}$$

e.g. single spin-$\frac{1}{2}$ particle in magnetic field:

$$\hat{H} = -\vec{\mu}_s \vec{B} = -\mu_s \frac{\hbar}{2}\begin{pmatrix}\hat{\sigma}_x \\ \hat{\sigma}_y \\ \hat{\sigma}_z\end{pmatrix} \cdot \begin{pmatrix}B_x \\ B_y \\ B_z\end{pmatrix} = -\frac{\mu_s \hbar}{2}\left(\hat{\sigma}_x B_x + \hat{\sigma}_y B_y + \hat{\sigma}_z B_z\right)$$

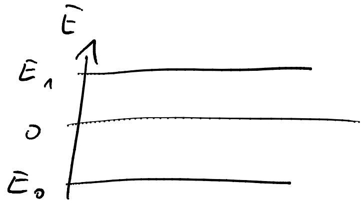
simplest case $\vec{B} = B_z \cdot \vec{e}_z$



$$\hat{H} = -\mu_s \frac{B\hbar}{2}\sigma_z$$

energy eigenstates: $\hat{H}|0\rangle = E_0|0\rangle$

$\hat{H}|1\rangle = E_1|0\rangle$



$$E_0 = -\left(-\frac{g_s e\hbar}{2m_e \cdot 2}\right)B \cdot (-1) = -\frac{g_s \mu_B B}{2}$$

$$E_1 = \frac{g_s \mu_B B}{2} \qquad \Delta E = E_1 - E_0 = g_s \mu_B B \hat{=} \hbar\omega_L$$

ω_L.. Larmor Frequency

What's the evolution, if $\vec{\mu}$ points in x-direction?

Larmor precession about z-axis!

$$|\psi(t)\rangle = \underbrace{e^{-\frac{i}{\hbar}\hat{H}t}}_{\hat{U}}|\psi(0)\rangle \qquad \text{for time-indep. } \hat{H}$$

$$-\frac{i}{\hbar}\hat{H} = i\frac{2\mu|B|}{2\hbar}\vec{n}\cdot\vec{\sigma} \qquad \text{from } \vec{B} = |B| \cdot \underset{\text{unit vector}}{\vec{n}} = |B| \cdot \begin{pmatrix}\sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta\end{pmatrix}$$

$$= i\frac{\omega}{2}\vec{n}\cdot\vec{\sigma}$$

$\omega = \frac{2\mu|B|}{\hbar}$... Larmor frequency

$$\hat{U} = e^{i\frac{\omega t}{2}\vec{n}\cdot\vec{\sigma}} = \cos\left(\frac{\omega t}{2}\vec{n}\cdot\vec{\sigma}\right) + i\sin\left(\frac{\omega t}{2}\vec{n}\cdot\vec{\sigma}\right) =$$

$$= \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}\underbrace{\left(\frac{\omega t}{2}\vec{n}\cdot\vec{\sigma}\right)^{2k}}_{\mathbb{1}\left(\frac{\omega t}{2}\right)^{2k}} + i\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\underbrace{\left(\frac{\omega t}{2}\vec{n}\cdot\vec{\sigma}\right)^{2k+1}}_{\left(\frac{\omega t}{2}\right)^{2k+1}\vec{n}\cdot\vec{\sigma}} =$$

$$= \mathbb{1}\sum\frac{(-1)^k}{(2k)!}\left(\frac{\omega t}{2}\right)^{k} + i\,\vec{n}\cdot\vec{\sigma}\sum\frac{(-1)^k}{(2k+1)!}\left(\frac{\omega t}{2}\right)^{2k+1}$$

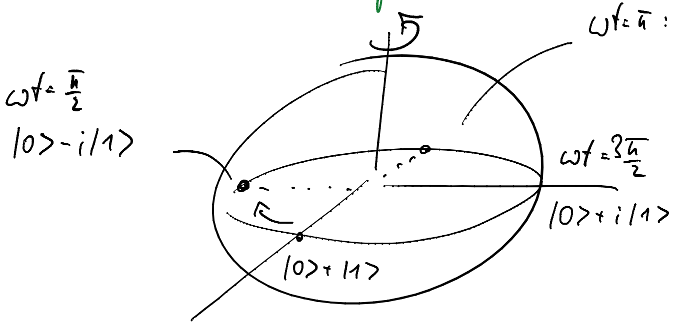
$$= \mathbb{1}\cdot\cos\frac{\omega t}{2} + i\,\vec{n}\cdot\vec{\sigma}\,\sin\frac{\omega t}{2}$$

$\vec{n} = n_z$:

$$= \hat{\mathbb{1}}\cos\frac{\omega t}{2} + i\,\hat{\sigma}_z\sin\frac{\omega t}{2} = \begin{pmatrix}\cos\frac{\omega t}{2} + i\sin\frac{\omega t}{2} & 0\\ 0 & \cos\frac{\omega t}{2} - i\sin\frac{\omega t}{2}\end{pmatrix} =$$

$$= \begin{pmatrix} e^{i\frac{\omega t}{2}} & 0\\ 0 & e^{-i\frac{\omega t}{2}}\end{pmatrix}$$

Representation on Bloch Sphere:



rotation about z-axis!

$$|\psi(t)\rangle = U_z|\psi(0)\rangle = \begin{pmatrix} e^{+i\omega t/2} & 0\\ 0 & e^{-i\omega t/2}\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} =$$

$$\hat{=} \left(e^{i\omega t/2}|0\rangle\langle 0| + e^{-i\omega t/2}|1\rangle\langle 1|\right)\left(|0\rangle + |1\rangle\right) =$$

$$= e^{i\omega t/2}|0\rangle + e^{-i\omega t/2}|1\rangle = e^{i\frac{\omega t}{2}}\left(|0\rangle + e^{-i\omega t}|1\rangle\right)$$

Question: In the experiment, how do you implement a phaseflip gate?

a) control the interaction time t

b) control the strength $|B|$ of the magnetic field

c) both

d) none of the above

Question: What is the evolution of the state $|0\rangle$ (on the Bloch sphere), when $\vec{B} = B_z$?

$\Uparrow \vec{B} = \vec{\mu}$

a) remains constant

b) obtains a phase $e^{-i\omega t/2}$

c) none of the above

I. 5. Quantum Measurement- Read Out

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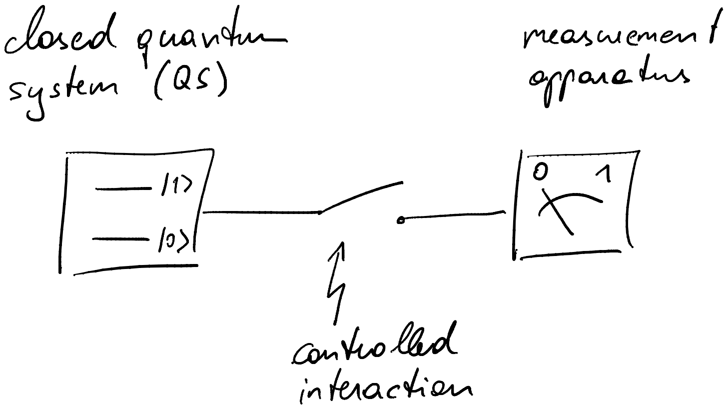
generic measurement setup



goal: faithful reconstruction of qubit state

What properties should an ideal measurement apparatus have?

- high ON/OFF ratio: no interaction between MA and QS when OFF, strong interaction when ON

- high fidelity of mapping of QS to MA state (no misidentifications)

$$|0\rangle \longrightarrow 0$$
$$|1\rangle \longrightarrow 1$$

- fast MA in comparison to decoherence / relaxation

- quantum non-demolition (QND): repeatability of measurement with same outcome

$$|\psi\rangle \,-\!\!-\, \boxed{\hat{M}} \,-\!\!-\, |\psi_m\rangle \; \cdots\cdots \; |\psi_m\rangle \,-\!\!-\, \boxed{\hat{M}} \,-\!\!-$$

outcome m   free evolution   same outcome m

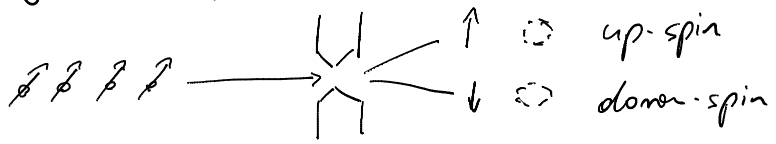
(Projective) Measurement

described by an observable (Hermitian operator) $\hat{M}$, e.g.

$\hat{S}_z$ ... spin along z-axis

$\hat{S}^2$ ... modulus square of spin

e.g. Stern-Gerlach apparatus



$$\hat{S}_z = \frac{\hbar}{2} \sigma_z$$

$$\sigma_z = 1 \cdot |1\rangle\langle 1| - 1 |0\rangle\langle 0| = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

can be decomposed into projection operators $\hat{P}_m$

$$\hat{M} = \sum m \hat{P}_m = \sum m |m\rangle\langle m|$$

↑ eigenvalues m

P_m corresponds to asking the question:

'Is the system in state $|m\rangle$?'

$$P_\uparrow |\uparrow\rangle = |\uparrow\rangle\underbrace{\langle\uparrow|\uparrow\rangle}_{1} = 1 |0\rangle \quad \text{answer 'yes'}$$

$$P_\uparrow |\downarrow\rangle = |\uparrow\rangle\underbrace{\langle\uparrow|\downarrow\rangle}_{0} = 0 \quad \text{answer 'no'}$$

Probability, to find outcome m is given by

$$p(m) = \langle\psi|P_m|\psi\rangle$$

State after measurement is given by

$$|\psi_m\rangle = \frac{P_m|\psi\rangle}{\sqrt{\langle\psi|P_m|\psi\rangle}} = \frac{P_m|\psi\rangle}{\sqrt{p_m}} \leftarrow \text{Normalization}$$

e.g. probability to find the value +1 when measuring σ_z in superpositionstate $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

$$p(+1) = \langle\psi|P_1|\psi\rangle = \langle\psi|1\rangle\langle 1|\psi\rangle =$$

$$= \frac{1}{2}(\langle 0| + \langle 1|)|1\rangle\langle 1|(|0\rangle + |1\rangle) = \frac{1}{2}$$

state after measurement is $\frac{P_1|\psi\rangle}{\sqrt{p_m}} = \frac{|1\rangle\langle 1|\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)}{1/\sqrt{2}} = |1\rangle$

=> normalized state after measurement

Expectation value: $\langle\hat{M}\rangle = \langle\psi|\hat{M}|\psi\rangle =$

$$= \langle\psi|\sum m P_m|\psi\rangle =$$

$$= \sum m \langle\psi|m\rangle\langle m|\psi\rangle =$$

$$= \sum m \underbrace{|\langle\psi|m\rangle|^2}_{p(m)}$$

e.g.: σ_z for groundstate

$$\langle 0|\sigma_z|0\rangle = \langle 0|\left(|1\rangle\langle 1| - |0\rangle\langle 0|\right)|0\rangle$$

$$= \overbrace{\langle 0|1\rangle}^{0}\langle 1|0\rangle - \overbrace{\langle 0|0\rangle}^{1}\langle 0|0\rangle$$

$$= -1$$

σ_z for excited state $\langle 1|\sigma_z|1\rangle = -1$

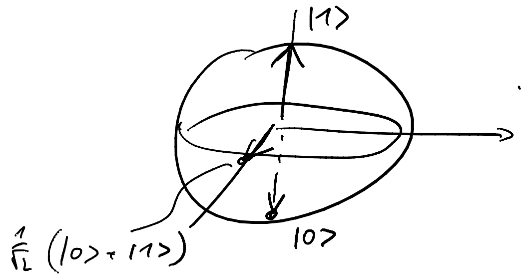
σ_z for superpos. state $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

$$\langle\psi|\sigma_z|\psi\rangle = \frac{1}{\sqrt{2}}\left(\langle 0| + \langle 1|\right)\sigma_z\left(|0\rangle + |1\rangle\right)\frac{1}{\sqrt{2}}$$

$$= \frac{1}{2}\left(\langle 0| + \langle 1|\right)\left(|1\rangle\langle 1| - |0\rangle\langle 0|\right)\left(|0\rangle + |1\rangle\right) =$$

$$= \frac{1}{2}(-1+1) = 0$$

σ_z measures projection of Bloch vector onto z-axis



Questions:

Is the measurement of σ_z sufficient to determine the state completely? (State tomography, see exercises)

Can you perform all required measurements simultaneously using a single particle? $\left([\hat{\sigma}_i, \hat{\sigma}_j] \neq 0 \quad \forall i \neq j ! \right)$