QSIT 2011 - Questions 3

16. March 2012, HIT F 13

1. **Bell states as a orthonormal basis**

   A Bell state is defined as a maximally entangled quantum state of two qubits. In such a state the qubits show perfect correlation even if they are spatially separated which cannot be explained without quantum mechanics. Show that the following four Bell states form a orthonormal basis for two qubits.

   (a) $|\Psi^+\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle_\mathrm{A}|1\rangle_\mathrm{B} + |1\rangle_\mathrm{A}|0\rangle_\mathrm{B}\right)$

   (b) $|\Psi^-\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle_\mathrm{A}|1\rangle_\mathrm{B} - |1\rangle_\mathrm{A}|0\rangle_\mathrm{B}\right)$

   (c) $|\Phi^+\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle_\mathrm{A}|0\rangle_\mathrm{B} + |1\rangle_\mathrm{A}|1\rangle_\mathrm{B}\right)$

   (d) $|\Phi^-\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle_\mathrm{A}|0\rangle_\mathrm{B} - |1\rangle_\mathrm{A}|1\rangle_\mathrm{B}\right)$

2. **Entanglement using CNOT gate**

   All the Bell states can be obtained by using one-qubit gates and a single two-qubit gate. Starting from the $|00\rangle$ state, show how to generate all four Bell states using NOT, Hadamard and CNOT gates.

$$\text{NOT:} \begin{array}{l} |0\rangle \to |1\rangle \\ |1\rangle \to |0\rangle, \end{array} \quad \text{H:} \begin{array}{l} |0\rangle \to \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \\ |1\rangle \to \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right), \end{array} \quad \text{CNOT:} \begin{array}{l} |0\rangle_\mathrm{A}|0\rangle_\mathrm{B} \to |0\rangle_\mathrm{A}|0\rangle_\mathrm{B} \\ |0\rangle_\mathrm{A}|1\rangle_\mathrm{B} \to |0\rangle_\mathrm{A}|1\rangle_\mathrm{B} \\ |1\rangle_\mathrm{A}|0\rangle_\mathrm{B} \to |1\rangle_\mathrm{A}|1\rangle_\mathrm{B} \\ |1\rangle_\mathrm{A}|1\rangle_\mathrm{B} \to |1\rangle_\mathrm{A}|0\rangle_\mathrm{B} \end{array}$$

3. **CNOT and CZ gates**

   In the previous exercise you obtained Bell states using a CNOT gate. They can also be obtained by a CZ gate, which follows from the universality of one-qubit and CNOT gates. Show that the CNOT gate can be obtained from a CZ gate using two Hadamard gates.

$$CZ = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

4. **Density matrix of a qubit entangled with another one**

The density operator formalism is used to describe a quantum system whose state is not completely known. Suppose a quantum system is in state $|\psi_i\rangle$ with respective probability p_i. The density operator for the system is defined as

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|.$$

Let us consider a system of two qubits, which is described by $|\psi_{\mathrm{AB}}\rangle$and let $\hat{O}$ be an observable of the qubit A. Then its expectation value is described by

$$\langle O \rangle = \mathrm{tr}[\rho_{\mathrm{A}} \hat{\mathrm{O}}],$$

where $\rho_{\mathrm{A}} = \mathrm{tr}_{\mathrm{B}}[\rho_{\mathrm{AB}}]$ is the reduced density operator of qubit A. For maximally entangled states such as the Bell states, ρ_{A} describes a maximally mixed state.

- Suppose that the system is in state $|\Psi^+\rangle$. What is the state of qubit A ignoring the state of qubit B?
- What is are the expectation values for σ_x^{A}, σ_y^{A}, σ_z^{A} for the $|\Psi^+\rangle$ state?