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Shor‘s algorithm: Order finding and factorization

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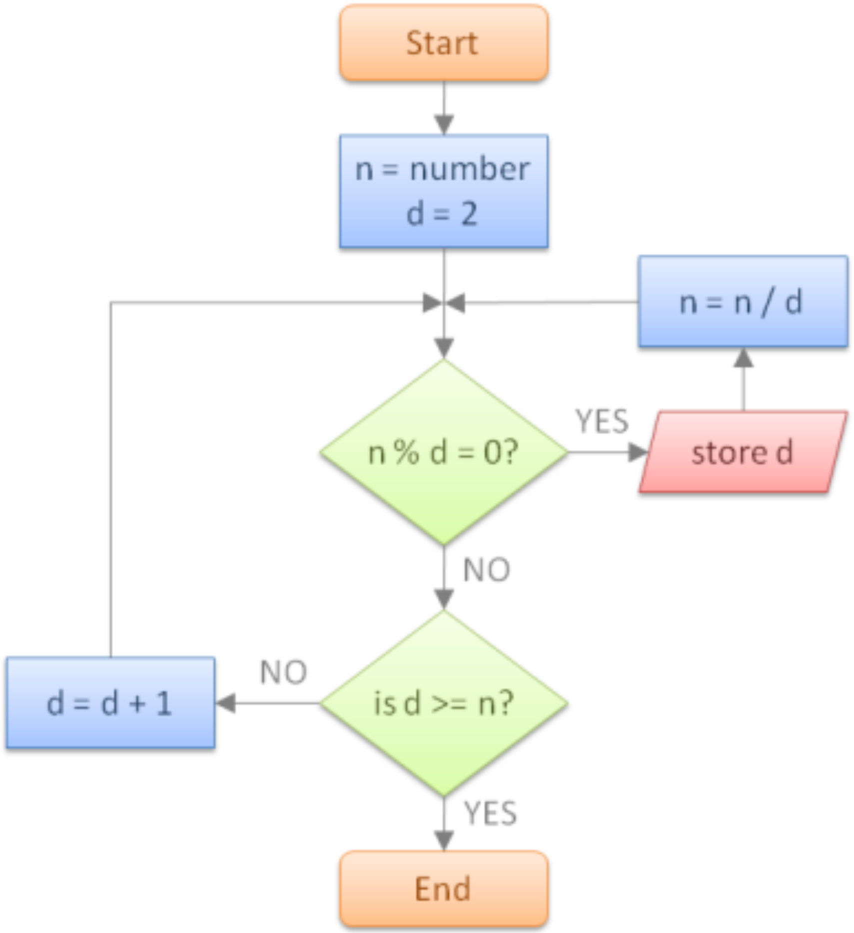
Outline

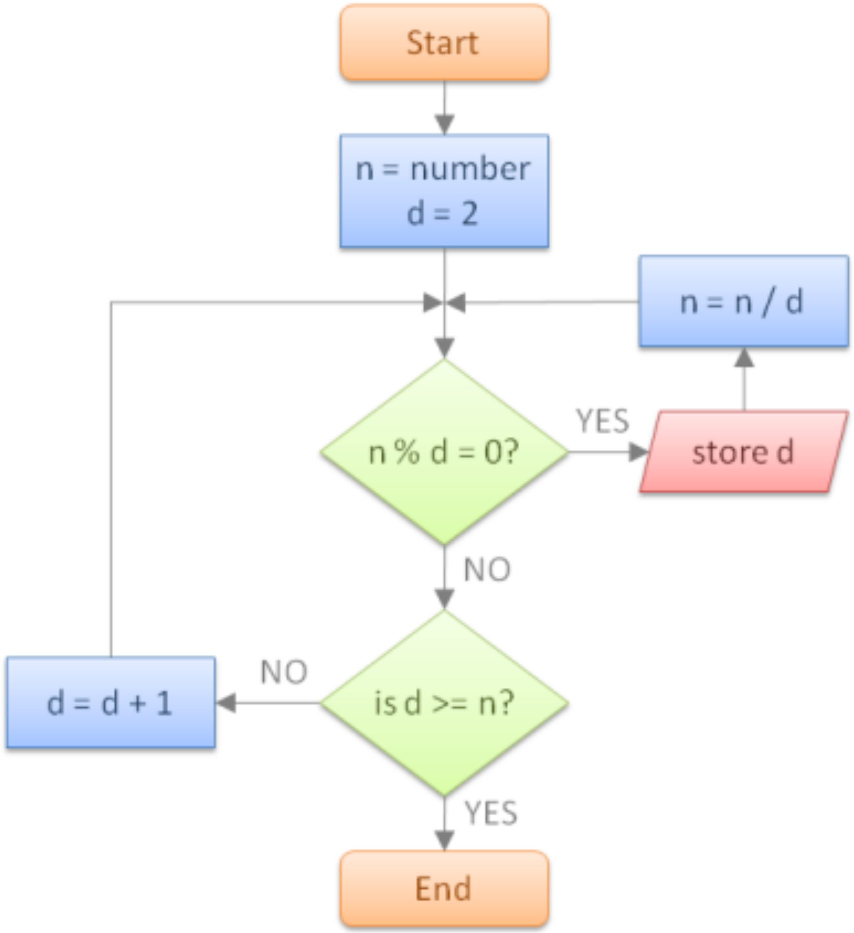
- Introduction
- Quantum Fourier Transform
- Phase Estimation
- Modular Exponentiation
- Order Finding
- Prime Factorization

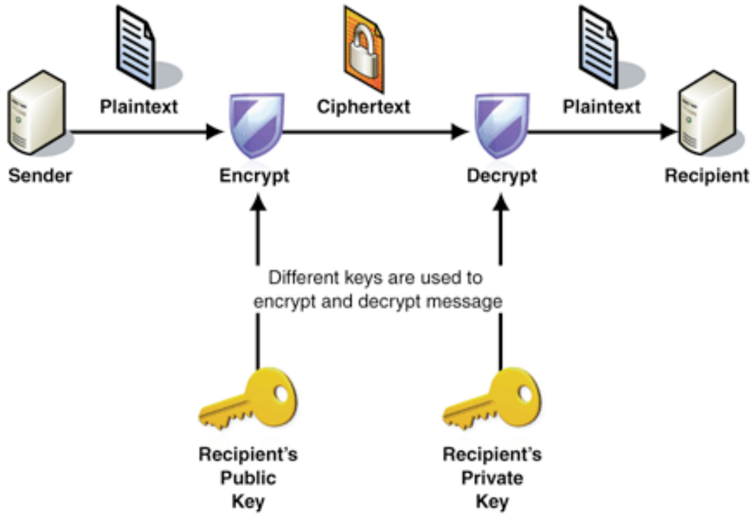
What is Shor‘s algorithm and why is it interesting?

Introduction



Introduction





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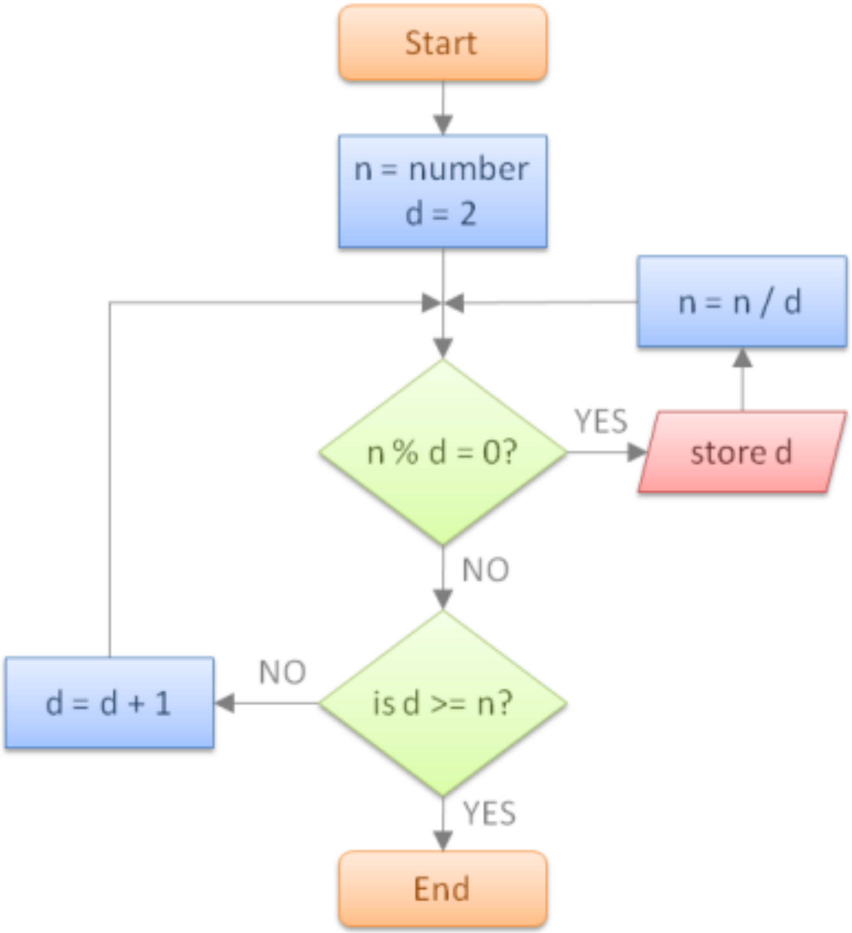
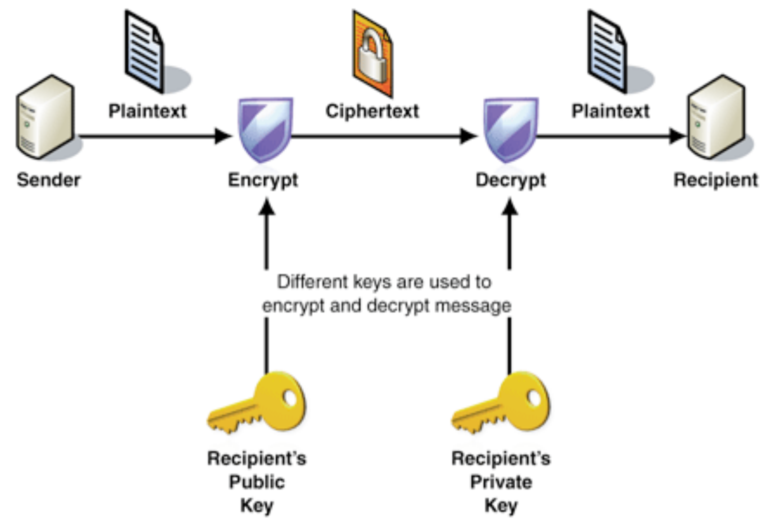


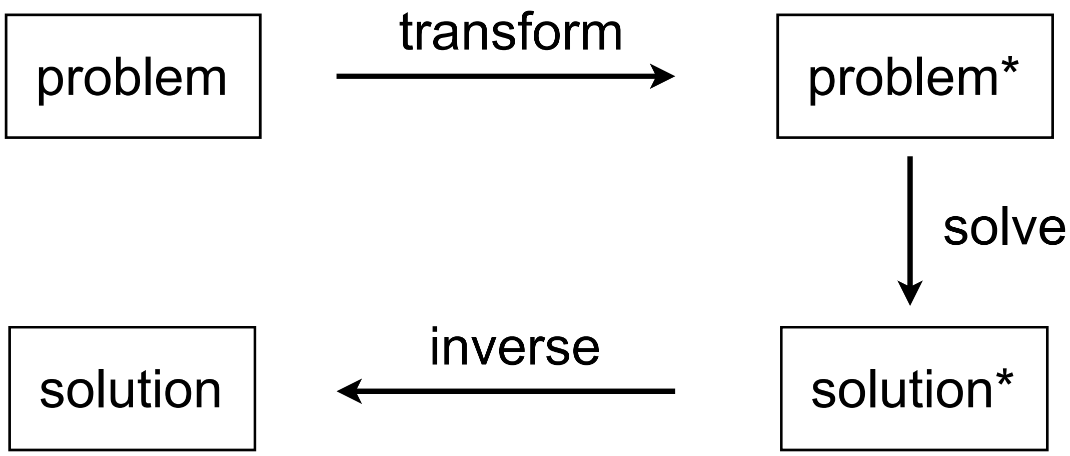


superpolynomial time on classical computers

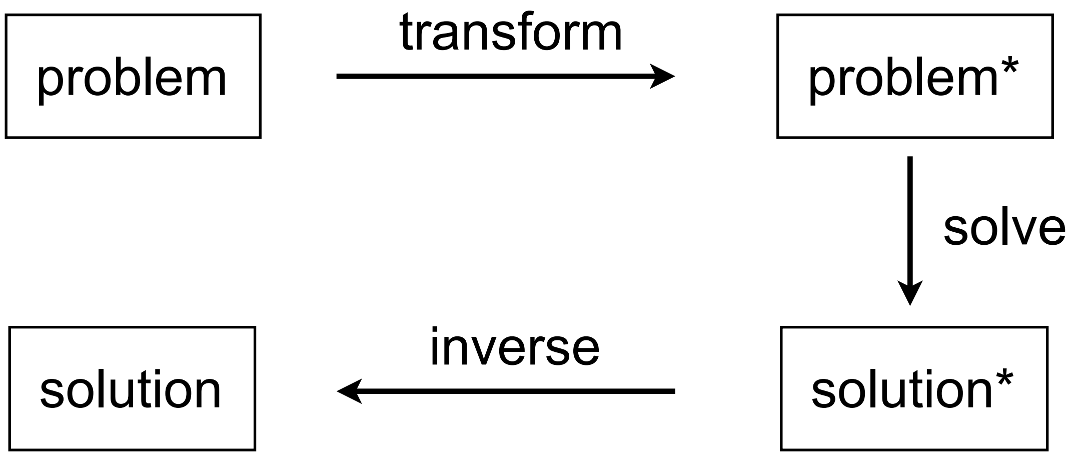
quantum polynomial time for Shor‘s algorithm!

Quantum Fourier Transform

Quantum Fourier Transform



Quantum Fourier Transform



discrete Fourier transform

Quantum Fourier Transform

classical

$$y_k \equiv \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} x_j e^{2\pi i j k / N}$$

Quantum Fourier Transform

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qm

$$|j\rangle \to \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} x_j e^{2\pi ijk/N} |k\rangle$$

Phase Estimation

Phase Estimation

- general procedure
- key for many quantum algorithms

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unitary operator	U
eigenvector	$\|u\rangle$
eigenvalue	$e^{2\pi i\varphi}$, unknown φ

Phase Estimation

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Phase Estimation

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Phase Estimation

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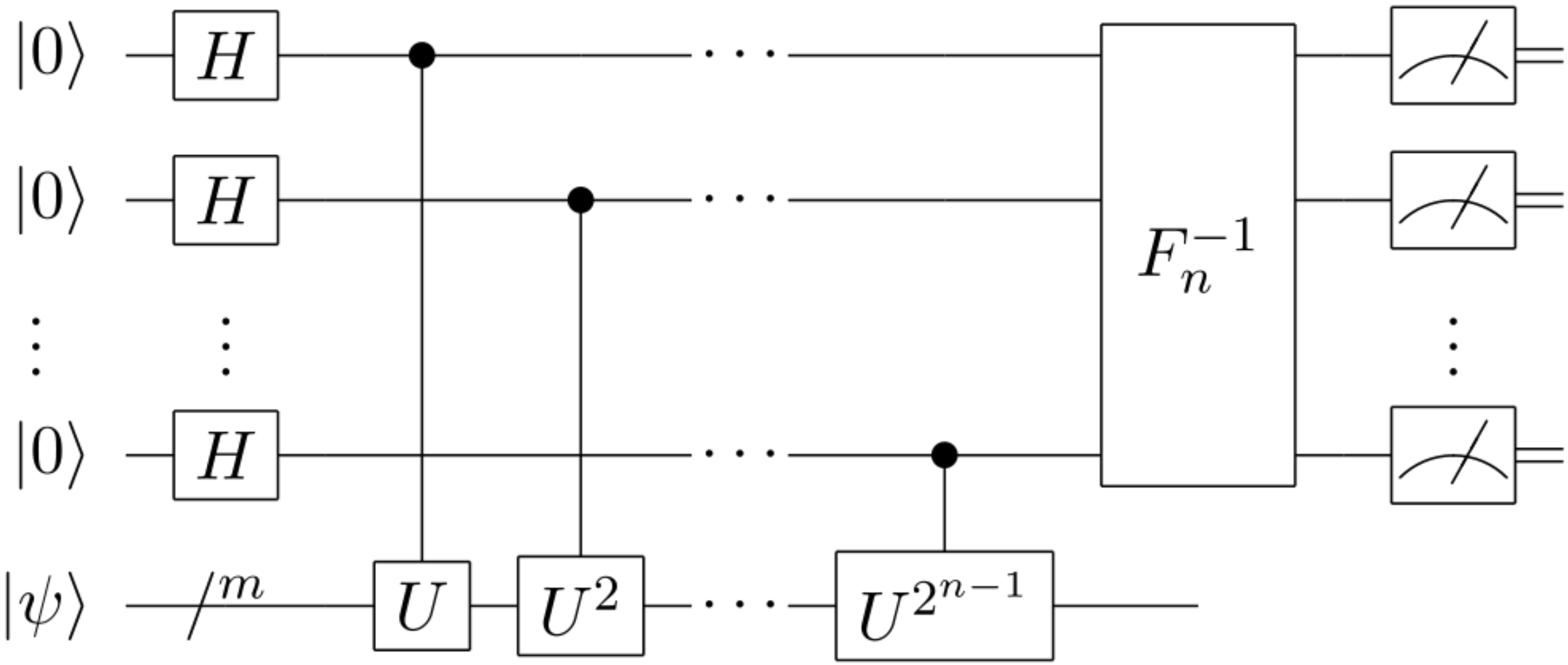
$$= \frac{1}{\sqrt{2^t}} \sum_{j=0}^{2^t-1} x_j e^{2\pi i j \varphi_u}\,|j\rangle\,|u\rangle$$

4. $\rightarrow |\varphi\rangle\,|u\rangle$ — apply inverse FT

Phase Estimation



Modular Exponentiation

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$$\frac{1}{\sqrt{2^t}} \sum_{j=0}^{t-1} x_j \left| j \right\rangle U^{2^j} \left| u \right\rangle$$

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$$\Big\downarrow \; \left|z\right\rangle \left|y\right\rangle$$

Modular Exponentiation

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Modular Exponentiation

$$U\,|y\rangle \;\equiv\; |xy(\mathrm{mod}N)\rangle$$

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$$U|y\rangle \equiv |xy(\mathrm{mod}N)\rangle$$

$$\frac{1}{\sqrt{2^t}}\sum_{j=0}^{t-1} x_j |z\rangle \left| x^{z_j 2^j} y(\mathrm{mod}N)\right\rangle$$

$$= \quad |z\rangle |x^z y(\mathrm{mod}N)\rangle$$

Order-finding

- Find least positive r for specified x and N such that:

$$x^r = 1 (\mathrm{mod}\ N)$$

- No classical algo exists polynomial in O(L)

$$L \equiv \lceil \log_2 (N) \rceil$$

Order-finding: Quantum algorithm

- Phase estimation applied to operator U

$$U|y\rangle \equiv |xy(\bmod N)\rangle \qquad y \in \{0,1\}^L$$

- Then eigenstates of U are:

$$|u_s\rangle \equiv \frac{1}{\sqrt{r}} \sum_{k=0}^{r-1} \exp\left[\frac{-2\pi i s k}{r}\right] |x^k \bmod N\rangle \qquad 0 \le s \le r-1$$

Order-finding: Quantum algorithm

$$|u_s\rangle \equiv \frac{1}{\sqrt{r}} \sum_{k=0}^{r-1} \exp\left[\frac{-2\pi i s k}{r}\right] |x^k \bmod N\rangle \qquad 0 \le s \le r-1$$

$$U|u_s\rangle = \frac{1}{\sqrt{r}} \sum_{k=0}^{r-1} \exp\left[\frac{-2\pi i s k}{r}\right] |x^{k+1} \bmod N\rangle$$

$$= \exp\left[\frac{2\pi i s}{r}\right] |u_s\rangle$$

Obtain estimate s/r using phase estimation procedure

Order r can be obtained with a little bit more work.

Order-finding: requirements

- Need efficient procedure for U for any

$$U|y\rangle \equiv |xy(\text{mod}\,N)\rangle$$

→satisfied by using modular exponentiation

- Must be able to prepare $|u_s\rangle$

→trickier, need r

exists clever fix for that

then we only obtain estimate $\varphi \approx s/r$

Order-finding: continued fraction expansion

- Now have estimate $\varphi \approx s/r$ would like to get r

Theorem: Suppose s/r is a rational number such that

$$\left|\frac{s}{r} - \varphi\right| \leq \frac{1}{2r^2}$$

Then s/r is a convergent of the continued fraction for φ.

→ Can use the continued fraction algorithm

Order-finding: continued fraction expansion

- The continued fraction algorithm

$$\frac{31}{13} = 2 + \frac{1}{2 + \frac{1}{1 + \frac{1}{1 + \frac{1}{2}}}}.$$

Can get s‘ and r‘ such that

$$\frac{s'}{r'} = \frac{s}{r}$$

➔ find correct r with probability > 1/4

Factoring algorithm

- Factoring can be reduced to order-finding

Theorem: *if x non trivial solution of*

$$x^2 = 1 (\mathrm{mod}\, N)$$

Then at least either gcd(x-1,N) or gcd(x+1,N) is a non-trivial factor of N. Can be computed using $O(L^3)$ *operations.*

Theorem: $N = p_1^{\alpha_1} \ldots p_m^{\alpha_m}$

x chosen at random $1 \leq x \leq N-1$ *and co-prime with N. r is order of x mod N.*

Then p(r is even and $x^{r/2} \neq -1 (\mathrm{mod}\, N)$*)* $\geq 1 - \frac{1}{2^m}$

Factoring algorithm

1. Determine if N trivially factorisable
2. Randomly choose x >0 and < N. if gcd(x,N)>1 return it
3. Order-finding to find r $x^r = 1(\mathrm{mod}\, N)$
4. If r even and $x^{r/2} \neq -1(\mathrm{mod}\, N)$ then compute gcd($x^{r/2}$-1,N) and gcd($x^{r/2}$+1,N)

➔Each of these 2 can be a nontrivial factor of N

If not: repeat 3-4

Conclusions

- Quantum algorithm factorizes in polynomial time

- Critical components:
  - Quantum FT
  - Modular exponentiation
  - Order finding