Lecture 5

Freitag, 21. Oktober 2011
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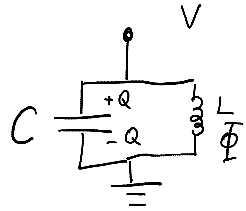
parallel LC oscillating circuit:



Voltage across oscillator:

$$V = \frac{Q}{C} = -\frac{d\Phi}{dt} = -L\frac{dI}{dt}$$

energy stored in the circuit: (Hamiltonian)

$$H = \underbrace{\frac{1}{2}\cdot CV^2}_{\substack{\text{capacitive} \\ \text{(electrostatic)} \\ \text{energy}}} + \underbrace{\frac{1}{2}LI^2}_{\substack{\text{inductive} \\ \text{(magnetic)} \\ \text{energy}}} = \frac{1}{2}\frac{Q^2}{C} + \frac{1}{2}\frac{\Phi^2}{L}$$

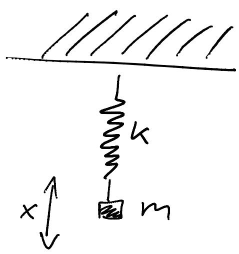
ϕ ... magnetic flux stored in the inductor

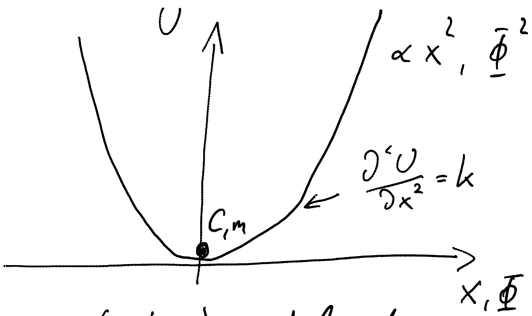
Q ... charge stored on the capacitor

mechanical harmonic oscillator:

$$H = \underbrace{\frac{p^2}{2m}}_{\text{kinetic}} + \underbrace{\frac{k}{2}x^2}_{\text{potential}}$$

energy





(virtual) particle of mass m (C) moving in potential

Characteristic quantities:

mechanical	electronic	HO
position x	flux Φ	canonical conjugate variables
momentum p	charge Q	
mass m	capacitance C	
spring constant k	inverse inductance $\frac{1}{L}$	
frequency $\omega = \sqrt{\frac{k}{m}}$	$\omega = \frac{1}{\sqrt{LC}}$	

Quantum harmonic oscillator

$x \rightarrow \hat{x}$ $\qquad$ $\Phi \rightarrow \hat{\Phi}$

$p \rightarrow \hat{p} = -i\hbar \frac{\partial}{\partial x}$ $\qquad$ $Q \rightarrow \hat{Q} = -i\hbar \frac{\partial}{\partial \Phi}$

Commutation relations:

$[\hat{x}, \hat{p}] = i\hbar$ $\qquad$ $[\hat{\Phi}, \hat{Q}] = i\hbar$ flux-charge $\quad / \cdot \frac{1}{\hbar}$

$\Leftrightarrow \left[2\pi \frac{\Phi}{\phi_0}, \frac{\hat{Q}}{2e}\right] = [\hat{\delta}, \hat{N}] = i$ phase-number

ϕ_0 ... magnetic flux quantum $\frac{h}{2e}$

Hamilton operator

$$H = \frac{\hat{Q}^2}{2C} + \frac{\hat{\Phi}^2}{2L} = -\frac{\hbar^2}{2C}\frac{\partial^2}{\partial \Phi^2} + \frac{1}{2L}\hat{\Phi}^2$$

creation and annihilation operators

$$\hat{a}^+ = \frac{1}{\sqrt{2\hbar Z_c}}\left(Z_c \hat{Q}^+ - i\hat{\Phi}^+\right) \quad \text{creation}$$

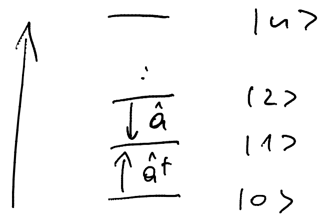
operator

$$\hat{a} = \frac{1}{\sqrt{2\hbar Z_c}}\left(Z_c \hat{Q} + i\hat{\Phi}\right) \quad \text{annihilation}$$

$Z_c = \sqrt{L/C}$, $\quad \omega = 1/\sqrt{LC}$ $\qquad$ impedance, resonance freq.

$\Rightarrow \hat{H} = \hbar\omega\left(\hat{a}^+\hat{a} - \frac{1}{2}\right)$ $\qquad$ $\hat{a}^+\hat{a}$ ... number operator energy level spectrum

equidistant level spacing



properties

$$\hat{a}^\dagger |n\rangle = \sqrt{n+1}\,|n+1\rangle$$

$$\hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle$$

$$\hat{a}^\dagger \hat{a}\,|n\rangle \equiv \hat{n}\,|n\rangle = n\,|n\rangle$$

$|n\rangle$ .. Fock state with n excitations

relation to $\hat{Q}$, $\hat{\Phi}$ & $\hat{V}$, $\hat{I}$ (inverse relations)

$$\hat{Q} = \sqrt{\frac{\hbar}{2Z_c}}\,(\hat{a}^\dagger + \hat{a}) \qquad \hat{\Phi} = i\sqrt{\frac{\hbar Z_c}{2}}\,(\hat{a}^\dagger - \hat{a})$$

or

$$\hat{V} = \frac{\hat{Q}}{C} = \sqrt{\frac{\hbar\sqrt{C}}{2C^2\sqrt{L}}}\,(\hat{a}^\dagger + \hat{a}) = \sqrt{\frac{\hbar\omega}{2C}}\,(\hat{a}^\dagger + \hat{a})$$

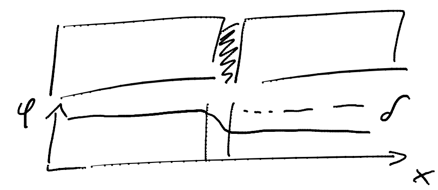
$$\hat{I} = \frac{\hat{\Phi}}{L} = i\sqrt{\frac{\hbar\omega}{2L}}\,(\hat{a}^\dagger - \hat{a}) \qquad \left[\omega = \frac{1}{\sqrt{LC}}\right]$$

Josephson Junction as Non-linear Inductor:

induction law: $V = -L\dot{I}$

Josephson relation: $I = I_0 \cdot \sin\delta$ $\boxed{dc}$

(supercurrent across junction)



$$V = \frac{\Phi_0}{2\pi}\dot{\delta} \qquad \boxed{ac}$$

(time-dependent phase if voltage is applied)

$$\dot I = I_0 \cdot \cos\delta \, \dot\delta \quad \Rightarrow \quad \boxed{V = \frac{\Phi_0}{2\pi I_0} \frac{1}{\cos\delta} \dot I \equiv L_J \dot I}$$

Josephson Inductance: $L_J = L_{J0} \frac{1}{\cos\delta} \rightarrow$ non-linearity

$L_{J0} = \frac{\Phi_0}{2\pi I_0}$ specific Josephson inductance

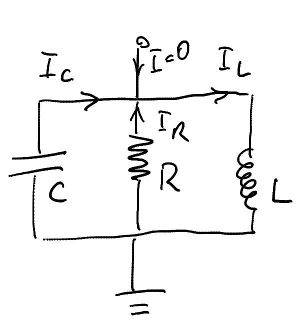
Josephson Energy:
$$E_J = \int V I \, dt' =$$
$$= \int \frac{\Phi_0 I_0}{2\pi} \dot\delta \sin\delta \, dt$$
$$= \int \frac{\Phi_0 I_0}{2\pi} \sin\delta \, d\delta = \frac{\Phi_0 I_0}{2\pi} \cos\delta$$
$$\boxed{= E_{J0} \cos\delta \quad \text{with } E_{J0} = \frac{\Phi_0 I_0}{2\pi}}$$

typical parameters: $I_0 = 100\,nA$

$\Rightarrow L_{J0} = \frac{\Phi_0}{2\pi I_0} \approx 3\,nH$ (~ 3mm of wire)

$\Rightarrow E_{J0} = \frac{\Phi_0 I_0}{2\pi} \approx 50\,GHz$

Dissipation:



$I_C + I_R = I_L$ (Kirchhoff-law)

- current through resistor: $I_R = \frac{V}{R}$
- displacement current:
$$I_C = \dot Q_C = C \dot V$$
- voltage across inductor
$$V = -L \dot I_L$$

$$\Rightarrow -C\dot{V} - \frac{V}{R} + I_L = 0$$

same voltage across elements:

$$\Rightarrow CL\ddot{I}_L + \frac{L}{R}\dot{I}_L - I_L = 0$$

$$\Rightarrow \boxed{\ddot{I}_L + \frac{1}{RC}\dot{I}_L - \frac{1}{LC} I_L = 0}$$

Differential equation for current through inductor

Solution: $I_L(t) = I_L(0)\, e^{\lambda t}$ $\quad {}_1\lambda_2 = \frac{1}{2LC}\left(-\frac{L}{R} \pm \sqrt{\left(\frac{L}{R}\right)^2 - 4LC}\right)$

($4LC \gg L/R$ ... underdamped oscillator)

$${}_1\lambda_2 = -\frac{1}{2RC} \pm i\frac{1}{\sqrt{LC}} = -\frac{\alpha}{2} \pm i\omega$$

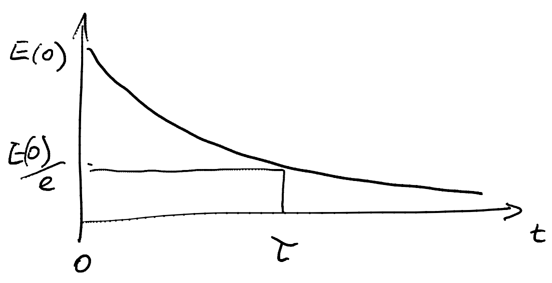
$\alpha = \frac{1}{RC} \equiv \frac{1}{\tau}$ decay constant

$\tau = RC$ decay time

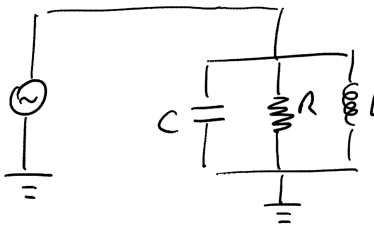
$\omega = \frac{1}{\sqrt{LC}}$ angular frequency

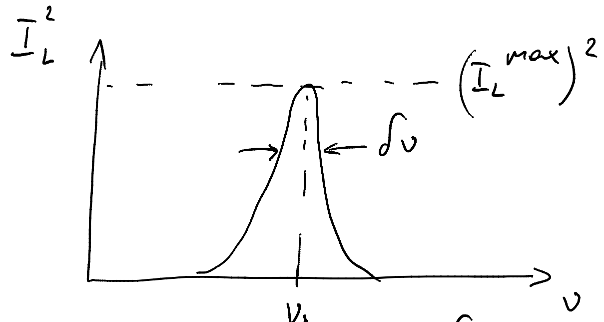
Energy decay rate:

$$E \propto L I_L^2 \propto e^{-\frac{1}{RC}t} = e^{-t/\tau}$$



spectral response: (driven damped oscillator)





Lorentzian line shape: $I_L^2(\nu) = (I_L^{max})^2 \cdot \dfrac{\frac{\delta\nu}{\pi}}{(\nu-\nu_1)^2 + \delta\nu^2}$

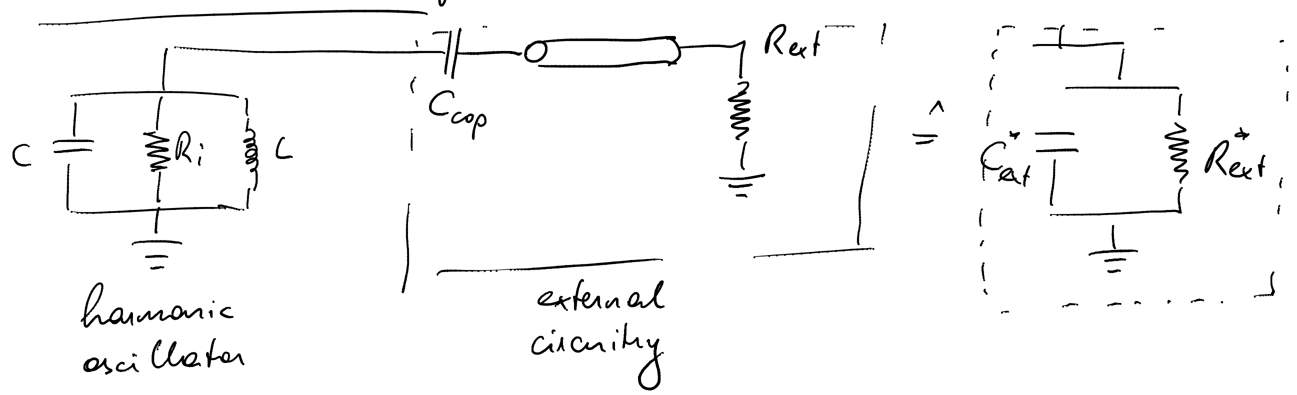
$\delta\nu = \frac{1}{2\pi\tau}$ ... full width at half maximum

Quality factor: $2\pi \dfrac{\text{Energy stored}}{\text{Energy lost per cycle}} \hat{=} \dfrac{\nu_1}{\delta\nu}$

$= 2\pi\nu_1 \cdot \tau = \underline{\omega_1 \cdot R \cdot C}$

in QM: excited state decay rate. $\Gamma_1 = \frac{1}{\tau} = \frac{1}{RC}$

Internal & external dissipation:



- total effective resistance (parallel capacitance) $\dfrac{1}{R_{tot}} = \dfrac{1}{R_{int}} + \dfrac{1}{R^*_{ext}}$ ⇒ external contribution to energy decay

— " — capacitance $C_{tot} = C_{int} + C^*_{ext}$

energy decay time $\tau = R_{tot} C_{tot}$