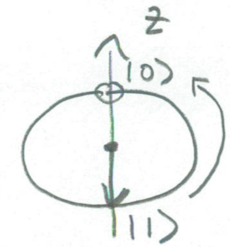
Single Qubit Gates

circuit representation

$$|\psi\rangle_{IN} \;\text{———}\; \boxed{\hat{O}} \;\text{———}\; |\psi\rangle_{OUT}$$

Work out the effect of the single qubit operations on some simple state on the Bloch sphere!

- $\hat{I} = \hat{\mathbb{1}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ Identity
- $\hat{X} = \hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ Bit flip

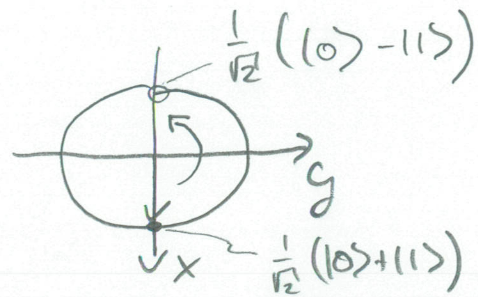


- $\hat{Y} = \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ Conjugate bit flip

$|0\rangle \to -i\,|1\rangle$

$|1\rangle \to i\,|0\rangle$

- $\hat{Z} = \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ Phase flip



Pauli matrices

The Hadamard Gate

generation of superposition from basis states

$$|0\rangle \;—\; \boxed{\hat{H}} \;—\; \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$

$$|1\rangle \;—\; \boxed{\hat{H}} \;—\; \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$$

- matrix representation

$$\hat{H} = \frac{1}{\sqrt{2}}\left(\hat{X} + \hat{Z}\right) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

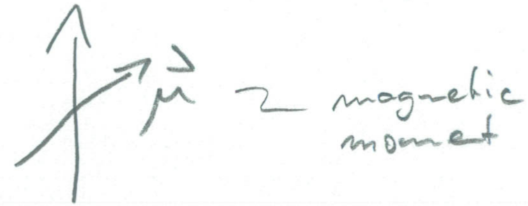
<u>Note:</u> - this gate is used in many quantum algorithms to prepare superposition states from basis states

Can you think of alternative ways to generate a superposition state?

Single Qubit Dynamics

magnetic field

$\vec{B} = (0, 0, B_z)$



spin 1/2 particle in external field

- Hamiltonian $\quad H = -\vec{\mu} \cdot \vec{B}$
- corresponding Operator $\quad \hat{H} = -\frac{g \mu_B B_z}{2} \hat{Z}$

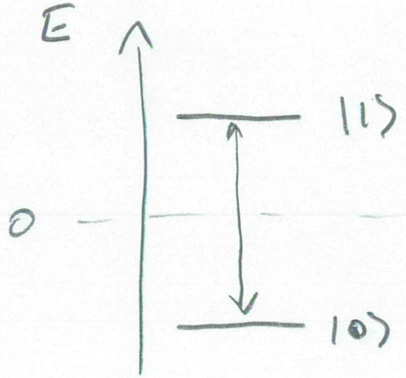
g : gyromagnetic ratio

μ_B : Bohr magneton

- time independent Schrödinger equation $\quad \hat{H} |\psi_i\rangle = E_i |\psi_i\rangle$
- eigenstates of $\hat{H}$ are $|0\rangle$ and $|1\rangle$

$$\hat{H} |0\rangle = E_0 |0\rangle$$
$$\hat{H} |1\rangle = E_1 |1\rangle$$

- energy level diagram



$$E_1 = \frac{g \mu_B B_z}{2}$$

$$E_0 = -\frac{g \mu_B B_z}{2}$$

$$\Delta E = g \mu_B B_z = \hbar \Omega_z = E_1 - E_0$$

- time dependent Schrödinger equation

$$i\hbar \frac{d}{dt} |\psi\rangle = \hat{H} |\psi\rangle$$

> How would you determine the dynamics of a system described by the operator $\hat{H}$?

- general solution for time independent $\hat{H}$

$$|\psi(t)\rangle = \exp\left(-\frac{i}{\hbar} \hat{H} t\right) |\psi(0)\rangle$$

with $$\exp(i\theta \hat{O}) = \cos\theta \, \hat{I} + i \sin\theta \, \hat{O}$$

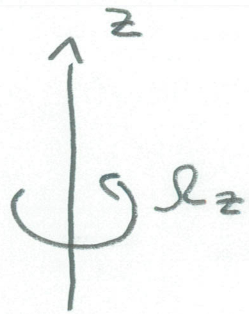
for Operators with $\hat{O}^2 = \hat{I}$ and $\theta \in \mathbb{R}$

e.g. for all Pauli matrices

- for spin 1/2 example

$$\hat{H} = -\frac{\hbar \Omega_z}{2} \hat{Z}$$

rotation about z-axis on Bloch sphere!



$$|\psi(t)\rangle = \exp\left(-\frac{i}{\hbar} \hat{H} t\right) |\psi(0)\rangle$$

$$= \left(\cos\frac{\theta_z}{2} \hat{I} + i \sin\frac{\theta_z}{2} \hat{Z}\right) |\psi(0)\rangle = R_z(\theta_z) |\psi(0)\rangle$$

with $\theta_z = \Omega_z t$

Dynamics of Superposition State

- initial state: $|\psi(0)\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$
- Hamilton operator $\hat{H} = -\frac{\hbar \Omega_z}{2} \underbrace{\left(|0\rangle\langle 0| - |1\rangle\langle 1|\right)}_{\hat{Z}}$
- final state

$$|\psi(t)\rangle = \exp\left(-\frac{i}{\hbar}\hat{H} t\right)|\psi(0)\rangle$$

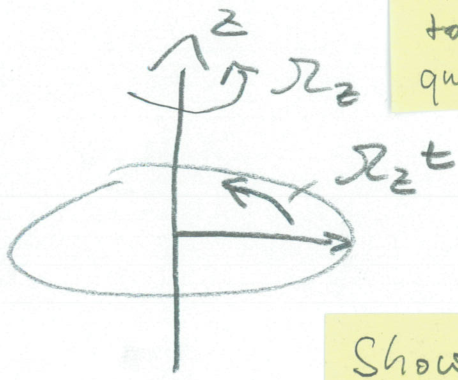
$$= \frac{1}{\sqrt{2}}\left(e^{i\frac{\Omega_z t}{2}}|0\rangle + e^{-i\frac{\Omega_z t}{2}}|1\rangle\right)$$

upto global phase $= \frac{1}{\sqrt{2}}\left(|0\rangle + \underbrace{e^{-i\Omega_z t}}_{e^{i\varphi}}|1\rangle\right)$

- on Bloch sphere

$\Theta = \frac{\pi}{2}$

$\varphi = -\Omega_z t$



Can you work out what dynamics the Hamiltonian $\hat{H}_x = -\frac{\hbar \Omega_x}{2}\hat{X}$ induces?

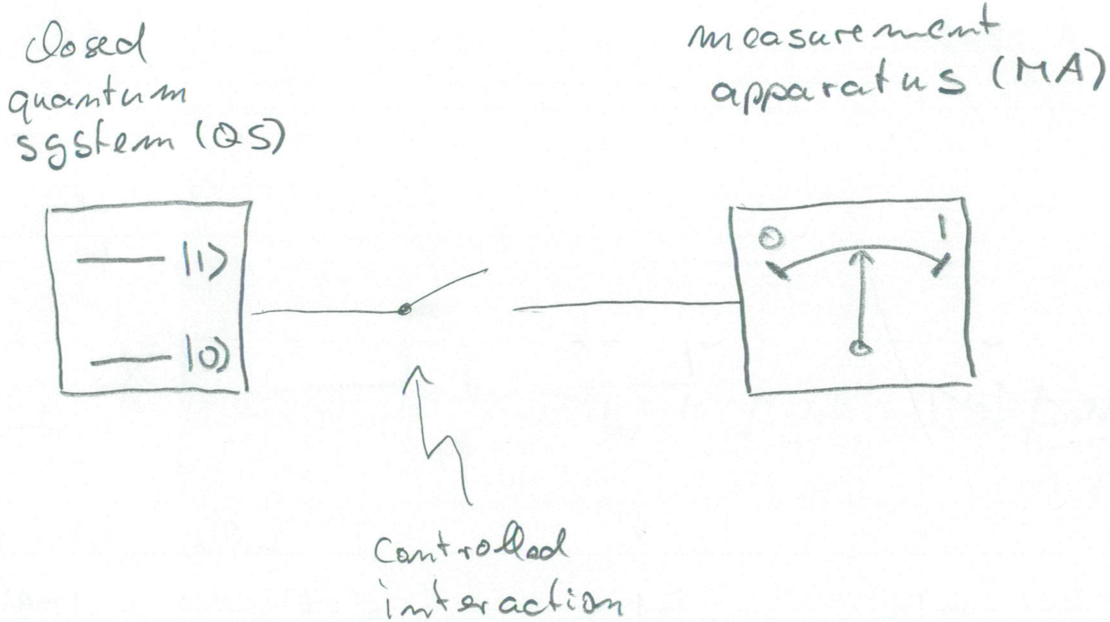
How is this useful for controlling the qubit state?

"Do the Bloch sphere dance to illustrate qubit dynamics!"

Show slides with other rotation operators.

Quantum Measurement

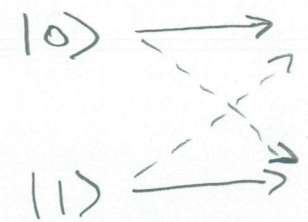
• generic set up



desired properties of measurement:

– ON/OFF: no interaction of MA with QS when OFF, strong interaction when ON

– high fidelity of mapping of QS state to MA state



– fast MA in comparison to coherence

– quantum non-demolition (QND): repeatability of measurement with same outcome

• goal: faithful reconstruction of qubit state

What properties do you suggest should an ideal measurement apparatus for a quantum bit have?

Measurement Postulate

- measurement result m with qubit in state $|\psi\rangle$ occurs with probability

$$P_m = \langle\psi| \hat{M}_m^\dagger \hat{M}_m |\psi\rangle$$

with a set of measurement operators $\{\hat{M}_m\}$ acting on the qubit states $|\psi\rangle$ that is complete

$$\sum_m P_m = 1 \iff \sum_m \hat{M}_m^\dagger \hat{M}_m = \hat{I}$$

- post measurement qubit state

$$|\psi'\rangle = \frac{\hat{M}_m |\psi\rangle}{\sqrt{P_m}}$$

Measurement of Qubit State in Computational Basis

- define measurement operators

$$\hat{M}_0 = |0\rangle\langle 0| = \begin{pmatrix}1\\0\end{pmatrix}(1\,0) = \begin{pmatrix}1&0\\0&0\end{pmatrix}$$

$$\hat{M}_1 = |1\rangle\langle 1| = \begin{pmatrix}0\\1\end{pmatrix}(0\,1) = \begin{pmatrix}0&0\\0&1\end{pmatrix}$$

complete $\sum_m \hat{M}_m^+ M_m = \hat{I}$

What do you think one can learn from a single measurement on a single qubit? What would you propose to do to learn more about the qubit state?

- example: measurement of $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

$$P_0 = \langle\psi|\hat{M}_0^+\hat{M}_0|\psi\rangle = \alpha^*\alpha = |\alpha|^2$$

$$P_1 = \langle\psi|\hat{M}_1^+\hat{M}_1|\psi\rangle = \beta^*\beta = |\beta|^2$$

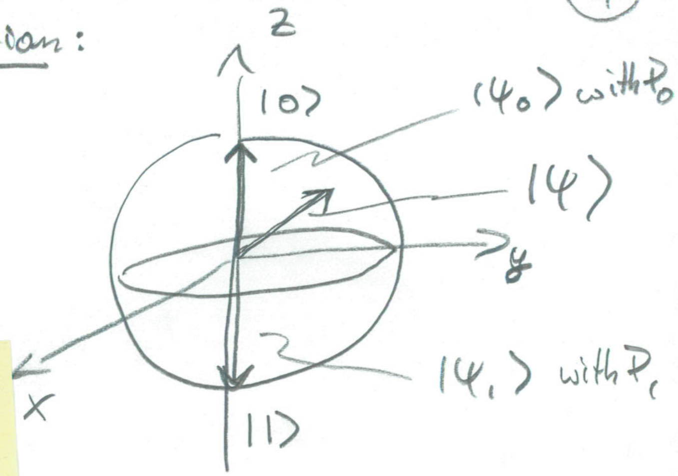
NOTE:
- single preparation of state $|\psi\rangle$ with single measurement $\hat{M}_m$ results in single outcome m with probability P_m
- to determine P_m, $|\psi\rangle$ has to be prepared and measured repeatedly (here determines $|\alpha|^2$ and $|\beta|^2$)
- full knowledge of state requires α, β to be known

- post measurement state

$$|\psi_0\rangle = \frac{\hat{M}_0|\psi\rangle}{\sqrt{P_0}} = \frac{\alpha}{|\alpha|}|0\rangle$$

$$|\psi_1\rangle = \frac{\hat{M}_1|\psi\rangle}{\sqrt{P_1}} = \frac{\beta}{|\beta|}|1\rangle$$

interpretation:



In your opinion does this type of measurement suffice to fully describe a qubit state?

- repeated measurement

$$P_{00} = \langle\psi_0|\hat{M}_0^\dagger\hat{M}_0|\psi_0\rangle = 1$$

probability of result of second measurement to be $m=0$ provided that first result was $m=0$

$$P_{01} = 0$$

$$P_{10} = 0$$

$$P_{11} = 1$$

What do you think could be reasons that measurement is not repeatable with same result?

NOTE: – any projective measurement should fulfill the above properties

PROBLEMS: – spontaneous emission of QS
– stimulated emission or absorbtion in QS due to MA
– misidentification of state by measurement apparatus