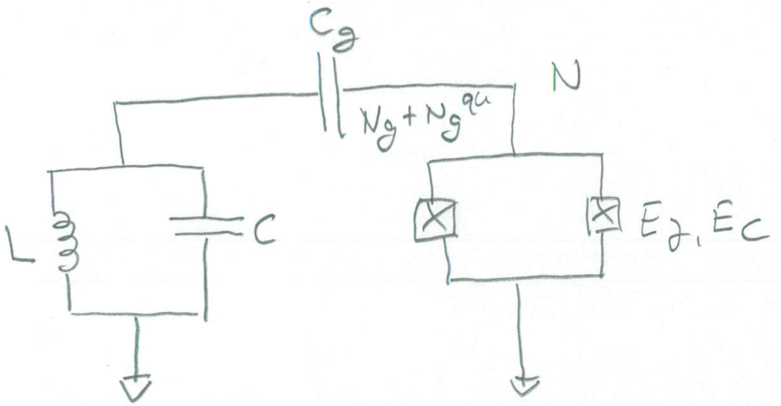
Jaynes - Cummings Hamiltonian in Circuit QED



$$\hat{H} = \underbrace{\frac{\hat{Q}^2}{2C} + \frac{\hat{\Phi}^2}{2L}}_{H.O.} + \frac{E_C}{2}\Big(\underbrace{1 - 2(N_g}_{=0 \text{ at } N_g = \frac{1}{2}} + N_g^{qu})\Big)\hat{\sigma}_z - \frac{E_J}{2}\hat{\sigma}_x$$

N_g^{qu} : quantum fluctuations of charge on capacitor C_g

$N_g = \frac{1}{2}$: consider charge degeneracy

quantum fluctuations of harmonic oscillator

$$\hat{H}_{HO} = \frac{1}{2} C \hat{V}^2 + \frac{1}{2} L \hat{I}^2$$

- consider only voltage fluctuations with operator $\hat{V}$

$$\hat{V} = \sqrt{\frac{\hbar\omega_r}{2C}}\,(\hat{a}^+ + \hat{a})$$

- variance of voltage fluctuations

$$\Delta V^2 = \langle 0 | \hat{V}^2 | 0 \rangle - \underbrace{\langle 0 | \hat{V} | 0 \rangle}_{\text{mean voltage} = 0 \text{ for } |n\rangle = |0\rangle}{}^2$$

$$= \frac{\hbar\omega_r}{2C} \underbrace{\langle 0 | (\hat{a}^+ + \hat{a})^2 | 0 \rangle}_{=0} = \frac{\hbar\omega_r}{2C}$$

- with quantum fluctuations of charge

$$N_g^{qu} = \frac{C_g}{2e} \hat{V}^{qu} = \frac{C_g}{2e} \sqrt{\frac{\hbar\omega_r}{2C}}\,(\hat{a}^+ + \hat{a})$$

Full Hamiltonian

- with change of basis $\hat{\sigma}_z \rightarrow \hat{\sigma}_x$ and $\hat{\sigma}_x \rightarrow -\hat{\sigma}_z$

$$\hat{H} = \hbar\omega_r(\hat{a}^+ a + \tfrac{1}{2}) + \frac{E_C}{2}\frac{C_g}{2e}\sqrt{\frac{\hbar\omega_r}{2C}}(\hat{a}^+ + \hat{a})\hat{\sigma}_x + \frac{E_J}{2}\hat{\sigma}_z$$

- and qubit raising and lowering operators $\hat{\sigma}^+, \hat{\sigma}^-$

$\hat{\sigma}_x = (\hat{\sigma}^+ + \hat{\sigma}^-)$ we find for the interaction

$$\frac{E_c}{2}\frac{C_g}{e}\sqrt{\frac{\hbar\omega_r}{2C}}\left(\underbrace{\cancel{\hat{a}^+\hat{\sigma}^+}}_{\to 0} + \hat{a}^+\hat{\sigma}^- + \hat{a}\hat{\sigma}^+ + \underbrace{\cancel{\hat{a}\hat{\sigma}^-}}_{\to 0}\right)$$

rotating wave approximation (RWA)

- with $E_c = \dfrac{(2e)^2}{2C_\Sigma}$ the full Hamiltonian reads

$$\hat{H} = \hbar\omega_r(\hat{a}^+\hat{a}) + \hbar g(\hat{a}^+\hat{\sigma}^- + \hat{a}\hat{\sigma}^+) + \frac{E_J}{2}\hat{\sigma}_z$$

with coupling constant $\hbar g = \dfrac{C_g}{C_\Sigma} 2e \sqrt{\dfrac{\hbar\omega_r}{2C}}$

where $\dfrac{2g}{2\pi}$ is the vacuum Rabi frequency