2.0 Basic Elements of a Quantum Information Processor

2.1 Classical information processing

2.1.1 The carrier of information

- binary representation of information as **bits** (Binary digITs).

- classical bits can take values **either** 0 **or** 1

- information is represented (and stored) in a physical system
  - for example, as a voltage level at the input of a transistor in a digital circuit

- in Transistor-Transistor-Logic (TTL)
  - "low" = logical 0 = 0 - 0.8 V
  - "high" = logical 1 = 2.2 - 5 V

- similar in other approaches
  - CMOS: complementary metal oxide semiconductor
  - ECL: emitter coupled logic

- information is processed by operating on bits using physical processes
  - e.g. realizing logical gates with transistors

2.1.2 Processing information with classical logic

- decomposition of logical operations in **single bit and two-bit operations**

truth table of operation

- trivial single bit logic gate: **Identity**

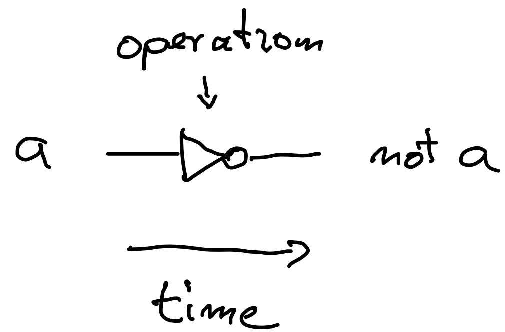
IN	OUT
1	1
0	0

- non-trivial single bit logic gate: **NOT**

IN	OUT
0	1
1	0

- circuit representation

input bit



output bit

- representation of time evolution of information
- each wire represents a bit and transports information in time
- each gate operation represented by a symbol changes the state of the bit

2.1.3 The universal two-bit logic gate

- logical operations between two bits: AND, OR, XOR, NOR ...
  - can all be implemented using NAND gates

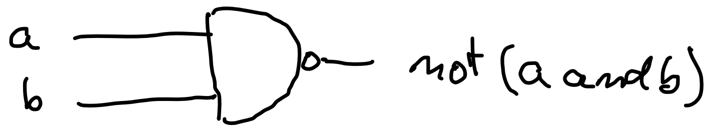
- Negation of AND : **NAND**

  AND followed by NOT

  truth table

IN	OUT
0 0	1
0 1	1
1 0	1
1 1	0

- circuit representation of the NAND gate:



Universality of the NAND gate:

- Any function operating on bits can be computed using NAND gates.
- Therefore NAND is called **a universal logic gate**.

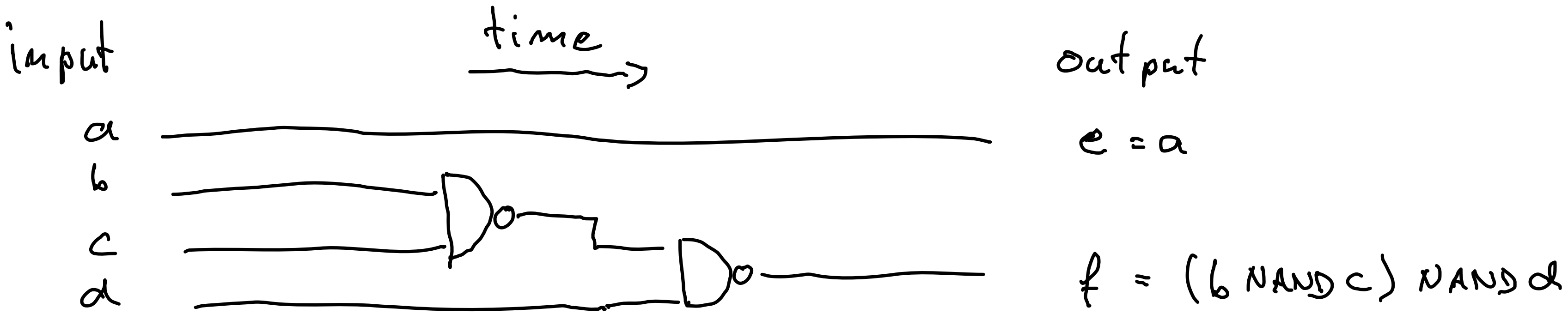
read: Nielsen, M. A. & Chuang, I. L., QC and QI, chapter 3, *Cambridge University Press,* (2000)

For quantum computation a set of universal gates has been identified

- single qubit operations and the CNOT gate form a universal set of gates for operation of a quantum computer

2.1.4 Circuit representation

- Any computable function can be represented as a circuit composed of universal gates acting on a set of input bits generating a set of output bits.



logical circuit computing a function

- properties of classical circuits representing a function
  - wires preserve states of bits
  - FANOUT: single input bit can be copied
  - additional working bits (ancillas) are allowed
  - CROSSOVER: interchange of the value of two bits
  - AND, XOR or NOT gates operate on bits
    - can be replaced by NAND gates using ancillas and FANOUT

Note:
- number of output bits can be smaller than number of input bits
  - information is lost, the process is not reversible
- no loops are allowed
  - the process has to be acyclic

- A similar circuit approach is useful to describe the operation of a quantum computer.
  - But how to make good quantum wires?
  - Can quantum information be copied?
  - How to make two-bit logic reversible?
  - What is a set of universal gates?

2.1.5 Conventional classical logic versus quantum logic

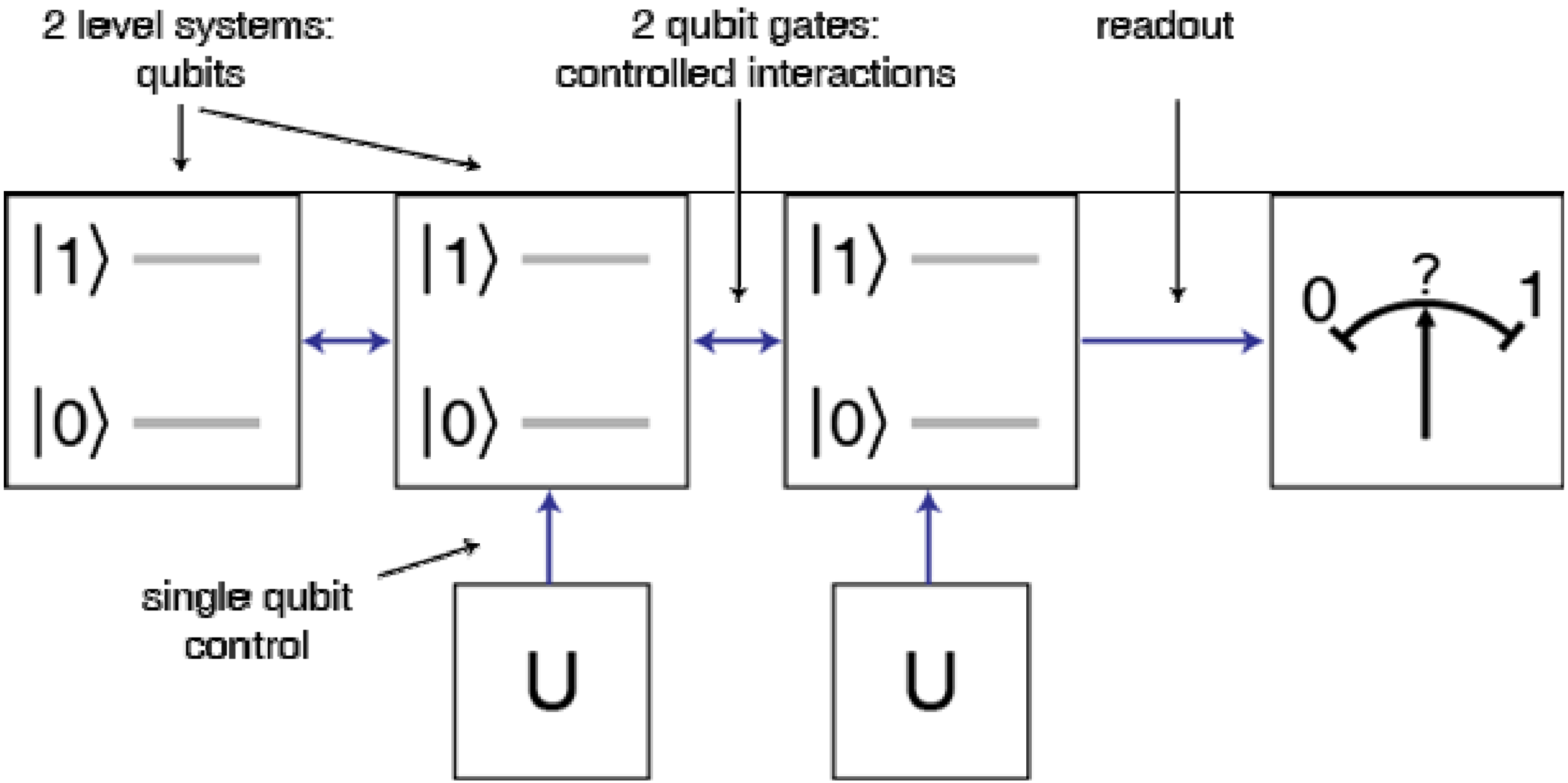
Conventional electronic circuits for information processing

- work according to the laws of **classical physics**
- quantum mechanics does not play a role in information processing

However:

- some devices used for information processing (LASERs, tunnel diodes, semiconductor heterostructures) operate using quantum mechanical effects on a microscopic level
- but macroscopic degrees of freedom (currents, voltages, charges) do usually not display quantum properties

2.2 Basic Components of a Generic Quantum Processor



2.2.1 The 5 DiVincenzo Criteria for Implementation of a Quantum Computer:

#1. A scalable physical system with well-characterized qubits.

#2. The ability to initialize the state of the qubits.

#3. Long (relative) decoherence times, much longer than the gate-operation time.

#4. A universal set of quantum gates.

#5. A qubit-specific measurement capability.

in the standard (circuit approach) to **quantum information processing** (QIP)

plus two criteria requiring the possibility to transmit information:

#6. The ability to interconvert stationary and mobile (or flying) qubits.

#7. The ability to faithfully transmit flying qubits between specified locations.

DiVincenzo, D., Quantum Computation, *Science* **270**, 255 (1995)

2.3 Quantum Bits

2.3.1 Classical Bits versus Quantum Bits

classical bit (**b**inary dig**it**)

- can take values 0 or 1

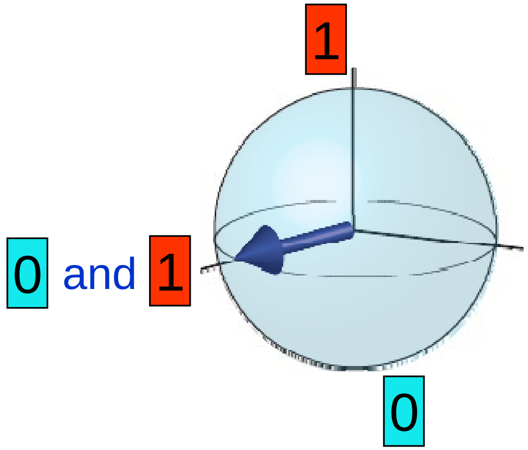
5 V 1

0 V 0

- realized e.g. as a voltage level 0 V or 5 V in a circuit

qubit (**qu**antum **bit**) [Schumacher '95]

- can take values 0 and 1 'simultaneously'



- realized as the quantum states of a physical system

- we will explore algorithms where the possibility to generate such states of the information carrying bit are essential

Schumacher, B., Quantum coding, *Phys. Rev. A* **51**, 2738-2747 (1995)

2.3.2 Definition of a Quantum Bit

Quantum **bit**s (qubits) are quantum mechanical systems with two distinct quantum mechanical states.

Qubits can be realized in a wide variety of physical systems displaying quantum mechanical properties.

- atoms, ions, molecules
- electronic and nuclear magnetic moments
- charges in semiconductor quantum dots
- charges and fluxes in superconducting circuits
- and many more ...

A suitable realization of a qubit should fulfill the so called **DiVincenzo criteria**.

Quantum Mechanical Description of a Qubit

A qubit has internal states that are represented as vectors in a 2-dimensional Hilbert space. A set of possible qubit (computational) basis states is:

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} ; \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \text{(Dirac notation)}$$

Quantum Mechanics Reminder:

QM postulate I: The quantum state of an isolated physical system is completely described by its state vector in a complex vector space with a inner product (a **Hilbert Space** that is). The state vector is a unit vector in that space.

Note:

This mathematical representation of a qubit allows us to consider its abstract properties independent of its actual physical realization.

2.3.3 Superposition States of a Qubit

A **quantum bit** can take values (quantum mechanical states) $|\psi\rangle$

$$|0\rangle , \quad |1\rangle$$

or both of them at the same time in which case the qubit is in a **superposition of states**

$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \quad \text{where} \quad \alpha, \beta \in \mathbb{C}$$

- when the state of a qubit is measured one will find

$$\begin{array}{lll} |0\rangle & \text{with probability} & |\alpha|^2 = \alpha \alpha^* \\ |1\rangle & \quad " & |\beta|^2 = \beta \beta^* \end{array}$$

- where the normalization condition is

$$\langle\psi|\psi\rangle = |\alpha|^2 + |\beta|^2 = 1$$

$$\text{with} \quad \langle\psi| = |\psi\rangle^\dagger = \alpha^* \langle 0| + \beta^* \langle 1| = (\alpha^*, \beta^*)$$

This just means that the sum over the probabilities of finding the qubit in any state must be unity.

Example: $|\psi\rangle = \frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle$ equal superposition state

2.3.4 Bloch sphere representation of qubit state space

alternative representation of qubit state vector useful for interpretation of qubit dynamics

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

$$= e^{i\gamma}\left[\cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle\right]$$

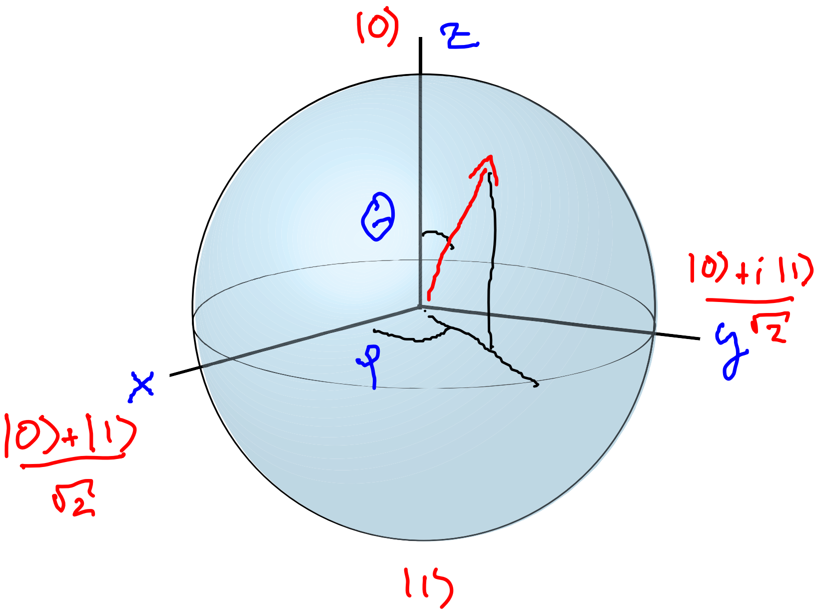
γ global phase factor

θ polar angle

φ azimuth angle

unit vector pointing at the surface of a sphere:

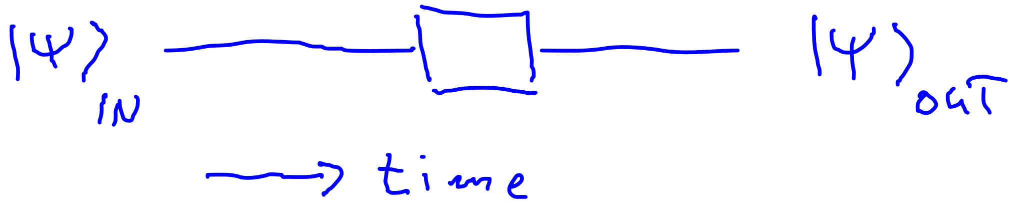
$$\vec{v} = (\cos\varphi\sin\theta, \sin\varphi\sin\theta, \cos\theta)$$



- ground state **|0>** corresponds to a vector pointing to the north pole
- excited state **|1>** corresponds to a vector pointing to the south pole
- equal superposition state **$(|0\rangle + e^{i\phi}|1\rangle)/2^{1/2}$** is a vector pointing to the equator

2.4 Single Qubit Logic Gates

2.4.1 Quantum circuits for single qubit gate operations



operations on single qubits:

$\boxed{X}$ bit flip $|0\rangle \rightarrow |1\rangle$; $|1\rangle \rightarrow |0\rangle$

$\boxed{Y}$ bit flip* $|0\rangle \rightarrow -i|1\rangle$; $|1\rangle \rightarrow i|0\rangle$

$\boxed{Z}$ phase flip $|0\rangle \rightarrow |0\rangle$; $|1\rangle \rightarrow -|1\rangle$

$\boxed{I}$ identity $|0\rangle \rightarrow |0\rangle$; $|1\rangle \rightarrow |1\rangle$

any single qubit operation can be represented as a rotation on a Bloch sphere

2.4.2 Pauli matrices

The action of the single qubit gates discussed before can be represented by Pauli matrices acting on the computational basis states:

bit flip (NOT gate) $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$; $X|0\rangle = |1\rangle$; $X|1\rangle = |0\rangle$

bit flip*(with extra phase) $Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$; $Y|0\rangle = i|1\rangle$; $Y|1\rangle = -i|0\rangle$

phase flip $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$; $Z|0\rangle = |0\rangle$; $Z|1\rangle = -|1\rangle$

identity $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$; $I|0\rangle = |0\rangle$; $I|1\rangle = |1\rangle$

all are unitary: $U = X, Y, Z, I$: $U^\dagger U = I$

exercise: calculate eigenvalues and eigenvectors of all Pauli matrices and represent them on the Bloch sphere

2.4.3 The Hadamard gate

a single qubit operation generating superposition states from the qubit computational basis states

$$|0\rangle \;—\boxed{H}—\; \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$

$$|1\rangle \;—\boxed{H}—\; \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$$

matrix representation of Hadamard gate:

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}}\left(X + Z\right) \quad ; \quad H^{\dagger} H = I$$

exercise: write down the action of the Hadamard gate on the computational basis states of a qubit.

2.5 Dynamics of Quantum Systems

2.5.1 The Schrödinger equation

QM postulate: The time evolution of a state **|ψ>** of a closed quantum system is described by the **Schrödinger equation**

$$i\hbar \frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle$$

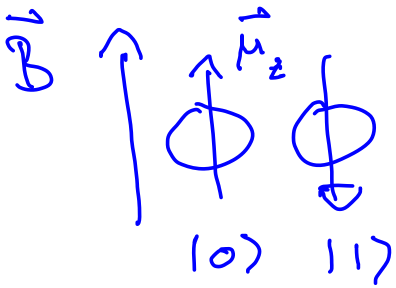
where ***H*** is the hermitian operator known as the **Hamiltonian** describing the closed system.

Reminder: A **closed quantum system** is one which does not interact with any other system.

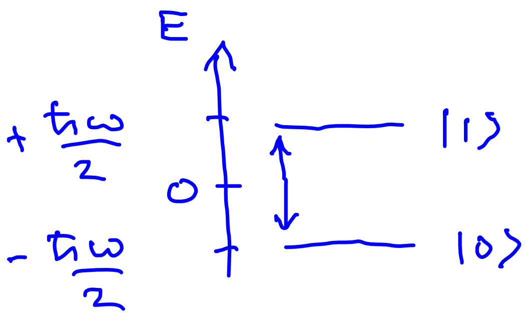
general solution for a time independent Hamiltonian ***H***:

$$|\psi(t)\rangle = \exp\left[\frac{-iHt}{\hbar}\right]|\psi(0)\rangle$$

example: e.g. electron spin in a field



energy level diagram:



Hamiltonian for spin 1/2 in a magnetic field: $H = -\frac{\hbar\omega}{2} Z$

$$H = -\frac{\hbar\omega}{2}\left(|0\rangle\langle 0| - |1\rangle\langle 1|\right)$$

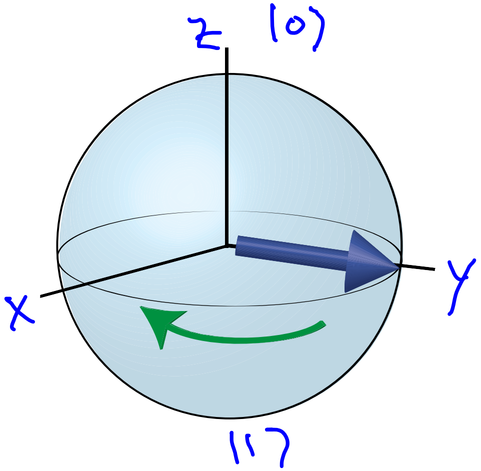
$$|\psi(0)\rangle = |0\rangle \longrightarrow |\psi(t)\rangle = e^{\frac{i\omega}{2}t}|0\rangle$$

$$|\psi(0)\rangle = |1\rangle \longrightarrow |\psi(t)\rangle = e^{-\frac{i\omega}{2}t}|1\rangle$$

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\cancel{e^{i\frac{\omega}{2}t}}\left(|0\rangle + e^{-i\omega t}|1\rangle\right)$$

interpretation of dynamics on the Bloch sphere:



$$|\psi\rangle = e^{i\gamma}\left(\cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle\right)$$

$$\Rightarrow \theta = \frac{\pi}{2}, \quad \varphi = -\omega t$$

this is a rotation around the equator of the Bloch sphere with **Larmor precession frequency** ω

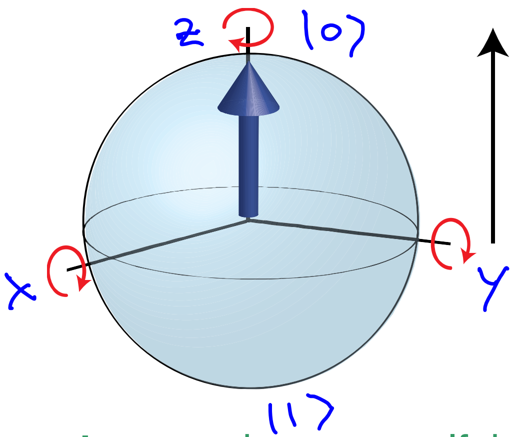
2.5.2 Rotation of qubit state vectors and rotation operators

when exponentiated the Pauli matrices give rise to rotation matrices around the three orthogonal axis in 3-dimensional space.

$$R_x(\theta) = e^{-i\theta X/2} = \cos\frac{\theta}{2} I - i \sin\frac{\theta}{2} X = \begin{pmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$

$$R_y(\theta) = e^{-i\theta Y/2} = \cos\frac{\theta}{2} I - i \sin\frac{\theta}{2} Y = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$

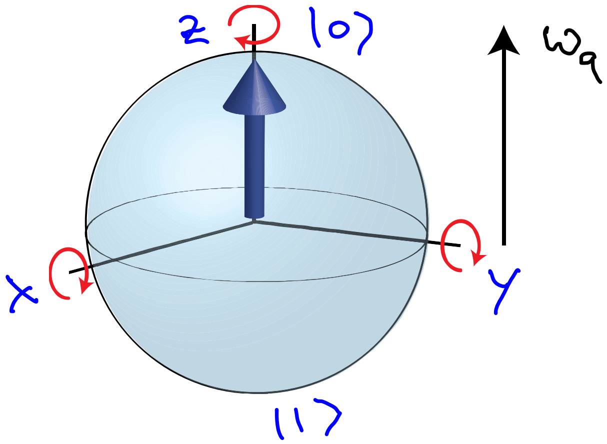
$$R_z(\theta) = e^{-i\theta Z/2} = \cos\frac{\theta}{2} I - i \sin\frac{\theta}{2} Z = \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix}$$



If the Pauli matrices **X**, **Y** or **Z** are present in the Hamiltonian of a system they will give rise to rotations of the qubit state vector around the respective axis.

exercise: convince yourself that the operators $\mathbf{R}_{x,y,z}$ do perform rotations on the qubit state written in the Bloch sphere representation.

2.5.3 Preparation of specific qubit states



initial state **|0>:**

prepare excited state by rotating around **x** or **y** axis:

X$_\pi$ pulse: $\Omega_x t = \pi$; $|0\rangle - \boxed{X_\pi} - |1\rangle$

Y$_\pi$ pulse: $\Omega_y t = \pi$; $|0\rangle - \boxed{Y_\pi} - i|1\rangle$

preparation of a superposition state:

X$_{\pi/2}$ pulse: $\Omega_x t = \frac{\pi}{2}$ $\quad |0\rangle - \boxed{X_{\pi/2}} - \frac{|0\rangle + |1\rangle}{\sqrt{2}}$

Y$_{\pi/2}$ pulse: $\Omega_y t = \frac{\pi}{2}$ $\quad |0\rangle - \boxed{Y_{\pi/2}} - \frac{|0\rangle + i|1\rangle}{\sqrt{2}}$

in fact such a pulse of chosen length and phase can prepare any single qubit state, i.e. any point on the Bloch sphere can be reached