

The Hadamard Gate

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generation of Superposition from basis states

$$|0\rangle \longrightarrow \boxed{\hat{H}} \longrightarrow \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle)$$

$$|1\rangle \longrightarrow \boxed{\hat{H}} \longrightarrow \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle)$$

• matrix representation

$$\hat{H} = \frac{1}{\sqrt{2}} \begin{pmatrix} \hat{X} + \hat{Z} \\ \hat{X} - \hat{Z} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Note: - this gate is used in many quantum algorithms to prepare superposition states from basis states

Can you think of alternative ways to generate a superposition state?

Single Qubit Dynamics

Spin 1/2 particle in external field

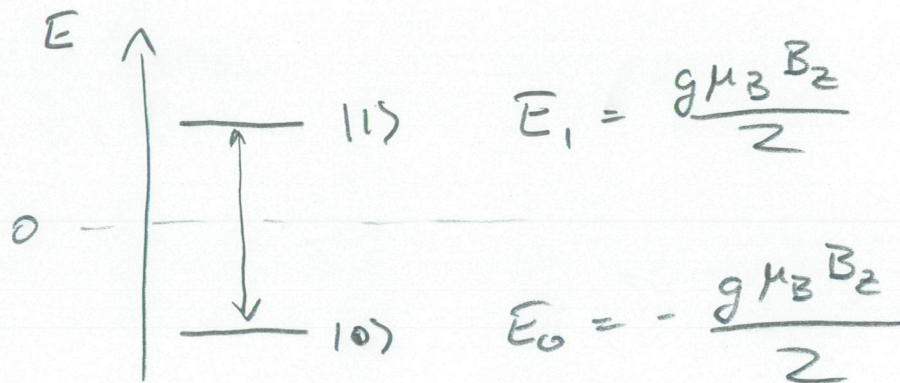
- Hamiltonian $H = -\vec{\mu} \cdot \vec{B}$

- Corresponding Operator $\hat{H} = -\frac{g\mu_B B_z}{2} \hat{z}$

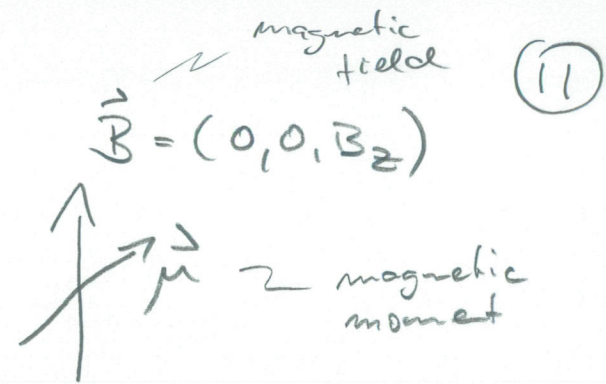
- time independent Schrödinger equation $\hat{H} |4_i\rangle = E_i |4_i\rangle$

- eigenstates of \hat{H} are $|0\rangle$ and $|1\rangle$
 $\hat{H} |0\rangle = E_0 |0\rangle$
 $\hat{H} |1\rangle = E_1 |1\rangle$

- Energy level diagram



$$\Delta E = g\mu_B B_z = \hbar \Omega_z = E_1 - E_0$$



g : gyromagnetic ratio
 μ_B : Bohr magneton

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- time dependent Schrödinger equation

$$i\hbar \frac{d}{dt} |\psi\rangle = \hat{H} |\psi\rangle$$

- general solution for time independent \hat{H}

$$|\psi(t)\rangle = \exp\left(-\frac{i}{\hbar} \hat{H} t\right) |\psi(0)\rangle$$

with

$$\exp(i\theta \hat{O}) = \cos \theta \hat{I} + i \sin \theta \hat{O}$$

for operators with $\hat{O}^2 = \hat{I}$ and $\theta \in \mathbb{R}$

e.g. for all Pauli matrices

- for spin $1/2$ example

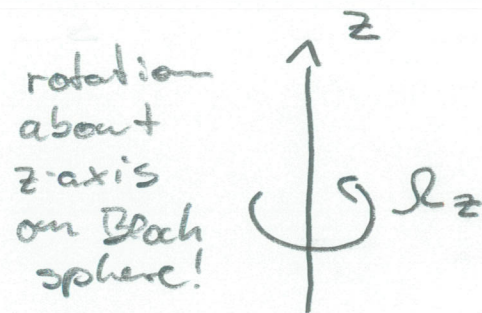
$$\hat{H} = -\frac{\hbar \mathcal{L}_z}{2} \hat{Z}$$

$$|\psi(t)\rangle = \exp\left(-\frac{i}{\hbar} \hat{H} t\right) |\psi(0)\rangle$$

$$= \left(\cos \frac{\theta_z}{2} \hat{I} + i \sin \frac{\theta_z}{2} \hat{Z}\right) |\psi(0)\rangle = R_z(\theta_z) |\psi(0)\rangle$$

with $\theta_z = \mathcal{L}_z t$

How would you determine the dynamics of a system described by the operator \hat{H} ?



Dynamics of Superposition State

- initial state
- Hamilton operator
- final state

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle)$$

$$\hat{H} = -\frac{\hbar J_z}{2} (|0\rangle\langle 0| - |1\rangle\langle 1|)$$

$$|\psi(t)\rangle = \exp\left(-\frac{i}{\hbar} \hat{H} t\right) |\psi(0)\rangle$$

$$= \frac{1}{\sqrt{2}} \left(e^{i \frac{J_z t}{2}} |0\rangle + e^{-i \frac{J_z t}{2}} |1\rangle \right)$$

upto global phase

$$= \frac{1}{\sqrt{2}} \left(|0\rangle + e^{-i J_z t} |1\rangle \right)$$

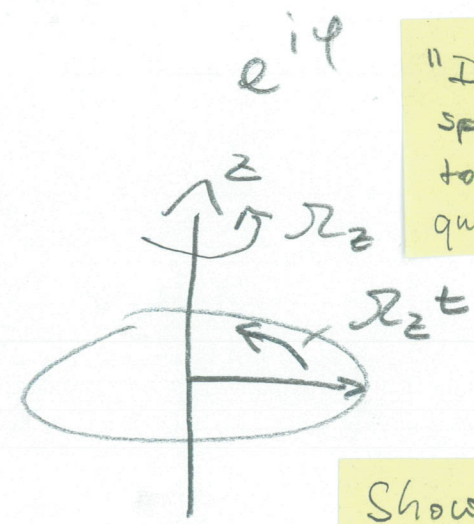
- on Bloch sphere

Can you work out what dynamics the Hamiltonian $\hat{H}_x = -\frac{\hbar J_x}{2} \hat{X}$ induces?

$$\Theta = \frac{\pi}{2}$$

$$\varphi = -J_z t$$

How is this useful for controlling the qubit state?

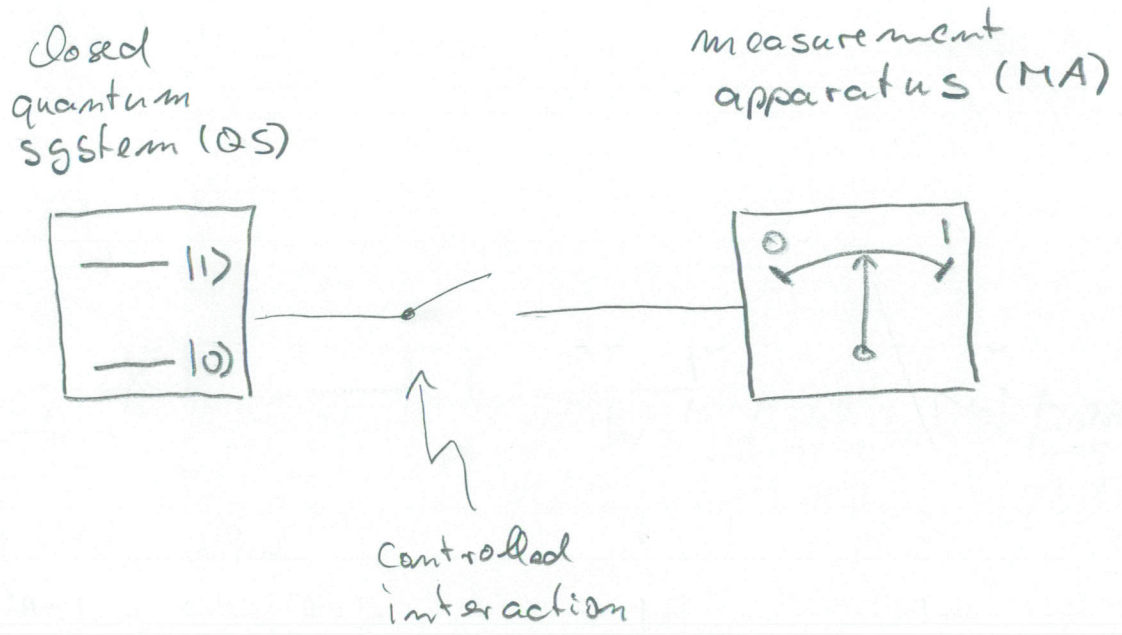


"Do the Bloch sphere dance to illustrate qubit dynamics!"

Show slides with other rotation operators.

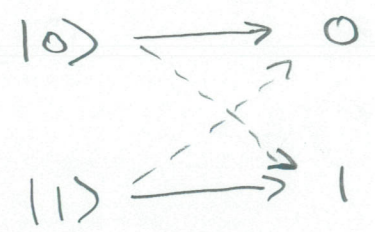
Quantum Measurement

- generic set up



desired properties of measurement:

- ON/OFF: no interaction of MA with QS when OFF, strong interaction when ON
- high fidelity of mapping of QS state to MA state



- goal: faithful reconstruction of qubit state

What properties do you suggest should an ideal measurement apparatus for a quantum bit have?

- fast MA in comparison to coherence
- quantum non-destruction (QND): repeatability of measurement with same outcome

Measurement Postulate

- Measurement result m with qubit in state $|\psi\rangle$ occurs with probability

$$P_m = \langle \psi | \hat{M}_m^\dagger \hat{M}_m | \psi \rangle$$

With a set of measurement operators $\{\hat{M}_m\}$ acting on the qubit states $|\psi\rangle$ that is complete

$$\sum_m P_m = 1 \quad \Leftrightarrow \quad \sum_m \hat{M}_m^\dagger \hat{M}_m = \hat{I}$$

- Post measurement qubit state

$$|\psi'\rangle = \frac{\hat{M}_m |\psi\rangle}{\sqrt{P_m}}$$

Measurement of Qubit State in Computational Basis

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- define measurement operators

$$\left. \begin{aligned} \hat{M}_0 &= |0\rangle\langle 0| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ \hat{M}_1 &= |1\rangle\langle 1| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned} \right\} \text{complete} \quad \sum_m \hat{M}_m^\dagger \hat{M}_m = \hat{I}$$

- example: measurement of $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

$$P_0 = \langle \psi | \hat{M}_0^\dagger \hat{M}_0 | \psi \rangle = \alpha^* \alpha = |\alpha|^2$$

$$P_1 = \langle \psi | \hat{M}_1^\dagger \hat{M}_1 | \psi \rangle = \beta^* \beta = |\beta|^2$$

What do you think one can learn from a single measurement on a single qubit? What would you propose to do to learn more about the qubit state?

NOTE:

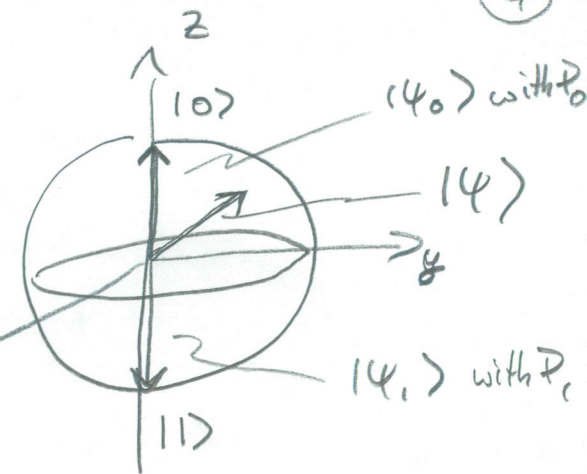
- Single preparation of state $|\psi\rangle$ with single measurement \hat{M}_m results in single outcome m with probability P_m
- to determine P_m , $|\psi\rangle$ has to be prepared and measured repeatedly (here determines $|\alpha|^2$ and $|\beta|^2$)
- full knowledge of state requires α, β to be known

• post measurement state

$$|\psi_0\rangle = \frac{\hat{M}_0 |\psi\rangle}{\sqrt{P_0}} = \frac{\alpha}{|\alpha|} |0\rangle$$

$$|\psi_1\rangle = \frac{\hat{M}_1 |\psi\rangle}{\sqrt{P_1}} = \frac{\beta}{|\beta|} |1\rangle$$

interpretation:



In your opinion does this type of measurement suffice to fully describe a qubit state?

• repeated measurement

$$P_{00} = \langle \psi_0 | \hat{M}_0^\dagger \hat{M}_0 | \psi_0 \rangle = 1$$

$$P_{01} = 0$$

$$P_{10} = 0$$

$$P_{11} = 1$$

What do you think could be reasons that measurement is not repeatable with same result?

probabilities of result of second measurement to be $m=0$ provided that first result was $m=0$

NOTE: - any projective measurement should fulfill the above properties

PROBLEMS:

- Spontaneous emission of QS
- Stimulated emission or absorption in QS due to MA
- misidentification of state by measurement apparatus

Multiple Qubit States and Entanglement

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Register of $n=2$ classical bits:

BIT A

BIT B

0
0
1
1

0
1
0
1

} 2^n different states

How many different states can two classical or two quantum mechanical bits be in?

register of $n=2$ quantum bits

QUBIT |A>

QUBIT |B>

|0>

|0>

|0>

|1>

|1>

|0>

|1>

|1>

} 2^n basis states

note: - only one state is realized at any given time

BUT: - quantum register can be in any superposition of basis states

formal description of general state of $n=2$ quantum register

$$|4\rangle = |A\rangle \otimes |B\rangle = |AB\rangle \text{ (according to 4th postulate)}$$

e.g. $|A\rangle = \alpha_A |0\rangle + \beta_A |1\rangle$; $|B\rangle = \alpha_B |0\rangle + \beta_B |1\rangle$

$$|4\rangle = \alpha_A \alpha_B |00\rangle + \alpha_A \beta_B |01\rangle + \beta_A \alpha_B |10\rangle + \beta_A \beta_B |11\rangle$$

with $\sum_{ij} |\alpha_{ij}|^2 = 1$ (normalization condition for probabilities)