QSIT 2009 - Questions 5

26. Oktober 2009

1. **Density matrix of a harmonic oscillator in thermal equilibrium**

   A statistical mixture of pure quantum states can be described by an incoherent sum of pure states denoted by the density matrix

   $$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|.$$

   p_i is the probability that the system is in the state $|\psi_i\rangle$, and the notation $|\psi_i\rangle\langle\psi_i|$ denotes the outer product of the state vector $|\psi_i\rangle$ and its conjugate transpose $\langle\psi_i|$.

   For example, if

   $$|\psi_1\rangle = (|0\rangle + \mathrm{i}|1\rangle)/\sqrt{2} = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathrm{i} \\ 1 \end{pmatrix},$$

   then

   $$\langle\psi_1| = (\langle 0| - \mathrm{i}\langle 1|)/\sqrt{2} = \frac{1}{\sqrt{2}} \begin{pmatrix} -\mathrm{i} & 1 \end{pmatrix}$$

   and

   $$|\psi_1\rangle\langle\psi_1| = \frac{1}{2} \begin{pmatrix} \mathrm{i} \\ 1 \end{pmatrix} \begin{pmatrix} -\mathrm{i} & 1 \end{pmatrix} = \begin{pmatrix} 1 & \mathrm{i} \\ -\mathrm{i} & 1 \end{pmatrix}.$$

   Similarly, if $|\psi_2\rangle = |0\rangle$ then

   $$|\psi_2\rangle\langle\psi_2| = \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

   If a source produces these state with equal probabilities ($p_1 = p_2 = 1/2$) the resulting state can be described by the statistical incoherent mixture

   $$\rho = \frac{1}{2}|\psi_1\rangle\langle\psi_1| + \frac{1}{2}|\psi_2\rangle\langle\psi_2| = \frac{1}{2}\left(\frac{1}{2}\begin{pmatrix} 1 & \mathrm{i} \\ -\mathrm{i} & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} \frac{1}{4} & \mathrm{i} \\ -\mathrm{i} & \frac{3}{4} \end{pmatrix}.$$

Let us now consider the thermal state of a harmonic oscillator. The eigenstates of a harmonic oscillator, which fulfill the Schrödinger equation $H|n\rangle = n\hbar\omega|n\rangle$, are denoted by $|0\rangle$, $|1\rangle \ldots, |n\rangle$ and correspond to states with n excitation quanta in the system. Neglecting the vacuum energy, the Hamiltonian $H = \hbar\omega_r n$, where n denotes the operator for the number of energy quanta, that is $n|0\rangle = 0, n|1\rangle = 1|1\rangle, n|2\rangle = 2|2\rangle, ...$

In thermal equilibrium with a heat bath at temperature T the probability p_n that the harmonic oscillator is excited to the nth state is given by the Boltzmann distribution

$$p_n = \frac{\exp[-E_n/(k_B T)]}{\sum_n \exp[-E_n/(k_B T)]} = \left(1 - \exp\left[-\frac{\hbar\omega}{k_B T}\right]\right) \exp\left[-\frac{n\hbar\omega}{k_B T}\right].$$

Find the density matrix which describes the equilibrium state of the harmonic oscillator.

2. **Mixed State decomposition**

   A source produces spin-1/2 particles polarized either in the state $|\uparrow\rangle$ or in the state $|\downarrow\rangle$ with equal probability. A different source produces also spin-1/2 particles, but polarized either in the state $|+\rangle = (|\uparrow\rangle + |\downarrow\rangle)/\sqrt{2}$ or in the state $|-\rangle = (|\uparrow\rangle - |\downarrow\rangle)/\sqrt{2}$, also with equal probability. Can you find a measurement which can distinguish these two sources?