

The Josephson Junction as a Non-Linear Inductor

(1)

induction law $V = -L \dot{I}$

Josephson equations $I = I_0 \sin \delta$ [dc] Josephson current

$$V = \frac{\Phi_0}{2\pi} \dot{\delta} \quad \text{[ac]}$$

with

$$\dot{I} = I_0 \cos \delta \dot{\delta}$$

follows

$$V = \frac{\Phi_0}{2\pi I_0} \frac{1}{\cos \delta} \dot{I} = L_J \dot{I}$$

Josephson inductance $L_J = L_{J0} \left(\frac{1}{\cos \delta} \right) \rightarrow$ non-linearity

$$L_{J0} = \frac{\Phi_0}{2\pi I_0} \quad \text{specific Josephson inductance}$$

Note: Phase difference δ in Josephson junction can be regarded as normalized magnetic flux

$$\delta = 2\pi \frac{\Phi}{\Phi_0}$$

Josephson Inductance and Josephson Energy

(2)

• Josephson energy

$$E_J = \int V I dt$$

$$= \int \frac{\Phi_0}{2\pi} \dot{\delta} I_0 \sin \delta dt$$

$$= \frac{\Phi_0 I_0}{2\pi} \cos \delta$$

$$= E_{J0} \cos \delta$$

$$\text{with } E_{J0} = \frac{\Phi_0 I_0}{2\pi}$$

• typical parameters:

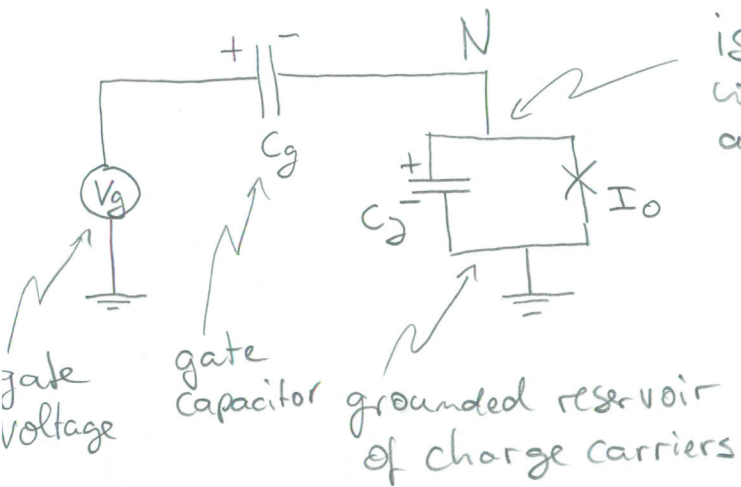
$$I_0 = 100 \text{ nA}$$

$$\Rightarrow L_{J0} = \frac{\Phi_0}{2\pi I_0} \approx 3 \text{ nH}$$

(~ 3 mm of wire)

$$\Rightarrow E_{J0} = \frac{\Phi_0 I_0}{2\pi} \approx 50 \text{ GHz}$$

The Cooper Pair Box Qubit



island on which charges are localized

$$N = \frac{Q}{2e}$$

discrete variable

number of Cooper pairs on island (with respect to charge neutrality)

$$N_g = \frac{C_g V_g}{2e}$$

polarization charge on gate capacitor

continuous variable

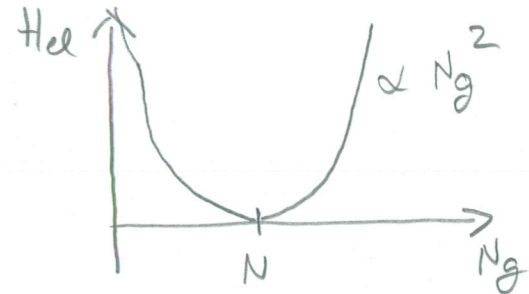
• Hamiltonian

$$H = \frac{Q^2}{2C} + \frac{\Phi^2}{2L}$$

electrostatic + magnetic energy

• electrostatic energy

$$H_{el} = \frac{Q^2}{2C} = \frac{(2e)^2 (N - N_g)^2}{2C_{\Sigma}}$$



with $C_{\Sigma} = C_j + C_g + \dots$ (stray capacitances)

total capacitance of island

and $E_C = \frac{(2e)^2}{2C_{\Sigma}}$ charging energy

How does the electrostatic energy depend on the gate voltage?

• magnetic energy

$$\begin{aligned}
H_{\text{mag}} &= -E_J \cos \delta = - \frac{\Phi_0 I_0}{2\pi} \overbrace{\cos \delta}^{\approx 1 - \frac{\delta^2}{2} + \dots} \\
&\approx - \frac{\Phi_0 I_0}{2\pi} \left(1 - \frac{1}{2} \left(\frac{\Phi}{\Phi_0} 2\pi \right)^2 + \dots \right) \\
&\approx \frac{1}{2} \frac{\Phi^2}{L J_0} \quad \text{(standard expression for mag. energy)}
\end{aligned}$$

• Cooper pair box Hamiltonian operator

$$\hat{H} = E_C (\hat{N} - N_g)^2 - E_J \cos \hat{\delta}$$

$\frac{1}{2}(e^{i\hat{\delta}} + e^{-i\hat{\delta}})$

with commutation relation $[\hat{\delta}, \hat{N}] = i$ for conjugate variables δ and N .

Transformation between bases:

$$|\delta\rangle = \frac{1}{\sqrt{2\pi}} \sum_N e^{iN\delta} |N\rangle$$

number states

• properties of phase $\hat{\delta}$ and number \hat{N} operators

$$[\hat{\delta}, \hat{N}] = i \quad \Rightarrow \quad e^{\pm i\hat{\delta}} |N\rangle = |N \pm 1\rangle$$

number operator $\hat{N} |N\rangle = N |N\rangle$

$$\sum_N |N\rangle\langle N| = \mathbb{1} \quad ; \quad \langle M|N\rangle = \delta_{M,N}$$

completeness orthogonality

Hamilton Operator of Cooper Pair Box in Charge Basis

$$\hat{H} = \sum_N \left(\underbrace{E_C (N - N_g)^2 |N\rangle\langle N|}_{\text{energy of charges on island}} - E_J/2 \underbrace{(|N\rangle\langle N+1| + |N+1\rangle\langle N|)}_{\text{energy to add or remove charges from island}} \right)$$

- solve time independent Schrödinger equation in discrete charge basis $|N\rangle$ to find energy eigenstates $|\psi\rangle$ of qubit

Show band diag. slides!

$$\hat{H} |\psi_m\rangle = E_m |\psi_m\rangle$$

- equivalent Hamilton operator in phase basis δ (continuous & periodic)

$$\hat{N} = \frac{\hat{Q}}{2e} = -i\hbar \frac{1}{2e} \frac{\partial}{\partial \Phi} = -i \frac{\partial}{\partial \delta}$$

$$\hat{H} = E_C \left(-i \frac{\partial}{\partial \delta} - N_g \right)^2 - E_J \cos \delta$$

\Rightarrow exact solutions for $\hat{H} \psi_m(\delta) = E_m \psi_m(\delta)$ are Mathieu functions