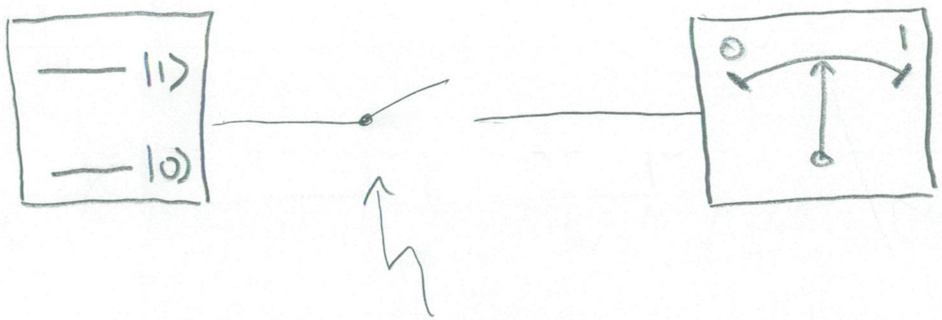
# Quantum Measurement

- generic set up

closed quantum system (QS)

measurement apparatus (MA)

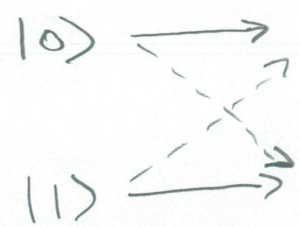



controlled interaction

- goal: faithful reconstruction of qubit state

**desired properties of measurement:**

- <u>ON / OFF</u>: no interaction of MA with QS when OFF, strong interaction when ON
- <u>high fidelity</u> of mapping of QS state to MA state




- <u>fast</u> MA in comparison to coherence
- quantum non-demolition (QND): repeatability of measurement with same outcome

> What properties do you suggest should an ideal measurement apparatus for a quantum bit have?

# Measurement Postulate

- measurement result $m$ with qubit in state $|\psi\rangle$ occurs with probability

$$P_m = \langle\psi| \hat{M}_m^\dagger \hat{M}_m |\psi\rangle$$

with a set of measurement operators $\{\hat{M}_m\}$ acting on the qubit states $|\psi\rangle$ that is complete

$$\sum_m P_m = 1 \iff \sum_m \hat{M}_m^\dagger \hat{M}_m = \hat{I}$$

- post measurement qubit state

$$|\psi'\rangle = \frac{\hat{M}_m |\psi\rangle}{\sqrt{P_m}}$$

# Measurement of Qubit State in Computational Basis

- define measurement operators

$$\left.\begin{aligned}\hat{M}_0 &= |0\rangle\langle 0| = \begin{pmatrix}1\\0\end{pmatrix}(1\;0) = \begin{pmatrix}1&0\\0&0\end{pmatrix}\\ \hat{M}_1 &= |1\rangle\langle 1| = \begin{pmatrix}0\\1\end{pmatrix}(0\;1) = \begin{pmatrix}0&0\\0&1\end{pmatrix}\end{aligned}\right\} \text{ complete } \sum_m \hat{M}_m^{+} M_m = \hat{I}$$

What do you think one can learn from a single measurement on a single qubit? What would you propose to do to learn more about the qubit state?

- example: measurement of $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

$$P_0 = \langle\psi|\hat{M}_0^{+}\hat{M}_0|\psi\rangle = \alpha^{*}\alpha = |\alpha|^2$$

$$P_1 = \langle\psi|\hat{M}_1^{+}\hat{M}_1|\psi\rangle = \beta^{*}\beta = |\beta|^2$$

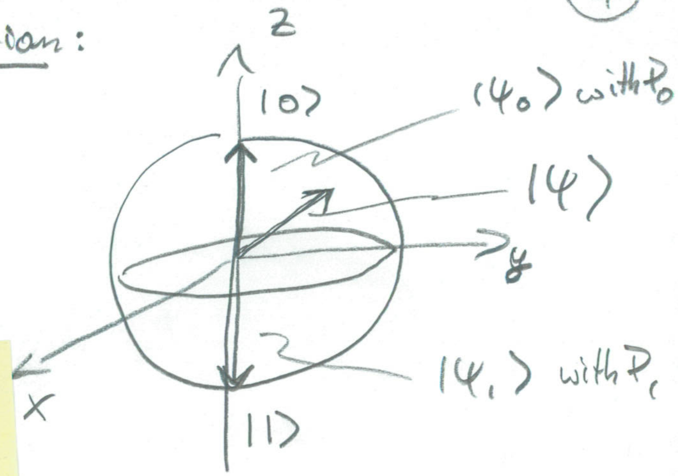
**NOTE:**
- single preparation of state $|\psi\rangle$ with single measurement $\hat{M}_m$ results in single outcome $m$ with probability $P_m$
- to determine $P_m$, $|\psi\rangle$ has to be prepared and measured repeatedly (here determines $|\alpha|^2$ and $|\beta|^2$)
- full knowledge of state requires $\alpha, \beta$ to be known

- post measurement state

$$|\psi_0\rangle = \frac{\hat{M}_0|\psi\rangle}{\sqrt{p_0}} = \frac{\alpha}{|\alpha|}|0\rangle$$

$$|\psi_1\rangle = \frac{\hat{M}_1|\psi\rangle}{\sqrt{p_1}} = \frac{\beta}{|\beta|}|1\rangle$$

interpretation:




In your opinion does this type of measurement suffice to fully describe a qubit state?

- repeated measurement

$$P_{00} = \langle\psi_0|\hat{M}_0^\dagger\hat{M}_0|\psi_0\rangle = 1$$

probability of result of second measurement to be $m=0$ provided that first result was $m=0$

$$P_{01} = 0$$

$$P_{10} = 0$$

$$P_{11} = 1$$

What do you think could be reasons that measurement is not repeatable with same result?

NOTE: - any projective measurement should fulfill the above properties of QS

PROBLEMS:
- Spontaneous emission in QS due to MA
- stimulated emission or absorbtion in QS due to MA
- misidentification of state by measurement apparatus

# Multiple Qubit States and Entanglement



How many different states can two classical or two quantum mechanical bits be in?

register of $n = 2$ classical bits:

| BIT A | BIT B |
|---|---|
| 0 | 0 |
| 0 | 1 |
| 1 | 0 |
| 1 | 1 |

$\left.\right\}$ $2^n$ different states

note: - only one state is realized at any given time

register of $n = 2$ quantum bits

| QUBIT $\|A\rangle$ | QUBIT $\|B\rangle$ |
|---|---|
| $\|0\rangle$ | $\|0\rangle$ |
| $\|0\rangle$ | $\|1\rangle$ |
| $\|1\rangle$ | $\|0\rangle$ |
| $\|1\rangle$ | $\|1\rangle$ |

$\left.\right\}$ $2^n$ basis states

BUT: - quantum register can be in any superposition of basis states

formal description of general state of $n=2$ quantum register

$$|\psi\rangle = |A\rangle \otimes |B\rangle = |AB\rangle \text{ (according to } 4^{th} \text{ postulate)}$$

e.g. $|A\rangle = \alpha_A |0\rangle + \beta_A |1\rangle$ ; $|B\rangle = \alpha_B |0\rangle + \beta_B |1\rangle$

$$|\psi\rangle = \alpha_A \alpha_B |00\rangle + \alpha_A \beta_B |01\rangle + \beta_A \alpha_B |10\rangle + \beta_A \beta_B |11\rangle$$

with $\sum_{ij} |\alpha_{ij}|^2 = 1$ (normalization condition for probabilities)

# Information Content of Many Qubit States

register of $n$ qubits:

- $2^n$ basis states
- general superposition state is described by $2^n$ complex coefficients

How would you best describe the state of $n=500$ qubits?
Is it at all possible?

consider $n = 500$ qubits

- need $2^{500} = 3 \times 10^{150}$ coefficients
  - $\Rightarrow$ larger than number of atoms in universe
  - $\Rightarrow$ impossible to store information about state classically

This is why it is difficult to simulate QM on a classical computer. But it would be natural to simulate QM on a quantum computer.

# Entangled Qubit States

<u>Definition</u>: An <u>entangled</u> state of a composite system is a state that <u>cannot</u> be written as a <u>product</u> state of the component systems.

product state: (example)

$$|\psi\rangle = |\psi_1 \psi_2\rangle \quad \text{with } |\psi_1\rangle = \alpha_1|0\rangle + \beta_1|1\rangle$$
$$|\psi_2\rangle = \alpha_2|0\rangle + \beta_2|1\rangle$$
$$= \alpha_1\alpha_2|00\rangle + \alpha_1\beta_2|01\rangle + \beta_1\alpha_2|10\rangle + \beta_1\beta_2|11\rangle$$

entangled state: (example)

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

How would you figure out, if this is an entangled state?

Does nature create such states?

How would you go about creating such a state?

$$\left.\begin{array}{l} \Rightarrow \alpha_1\alpha_2 = \frac{1}{\sqrt{2}} \wedge \beta_1\beta_2 = \frac{1}{\sqrt{2}} \\ \Rightarrow \alpha_1\beta_2 \neq 0 \wedge \alpha_2\beta_1 \neq 0 \end{array}\right\}$$ i.e. not a product state

<u>Questions</u>:

- How are such states created?
- What are their properties?

# Correlations of Entangled States



Measurement of individual qubit states in an entangled pair

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

What would you think is the result of a measurement of the state of both qubits?

- measure ground state $|0\rangle$ of first qubit (1)

$$\underset{\substack{\uparrow\\ \text{qubit}}}{P_1}\underset{\substack{\uparrow\\ \text{state}}}{(0)} = \langle\psi|\,(M_0\otimes I)^\dagger \underbrace{(M_0\otimes I)}_{\text{tensor products of individual qubit measurement operators}}|\psi\rangle = \frac{1}{2}$$

- post measurement state

$$|\psi'\rangle = \frac{(M_0\otimes I)\,|\psi\rangle}{\sqrt{P_1(0)}} = |00\rangle$$

- measure ground state $|0\rangle$ of second qubit (2) given that first one was measured in state $|0\rangle$.

$$P_2(0) = \langle\psi'|\,(I\otimes M_0)^\dagger (I\otimes M_0)\,|\psi'\rangle = 1$$

⇒ The outcomes of the measurements of both qubit states are 100 % correlated. Such correlations are impossible in a classical system (compare with Bell inequalities)

# Entanglement as a New Resource

Transmit two bits of classical information by sending one qubit between two parties <u>Alice</u> and <u>Bob</u>: <u>Super Dense Coding</u>

How is this better than classical?

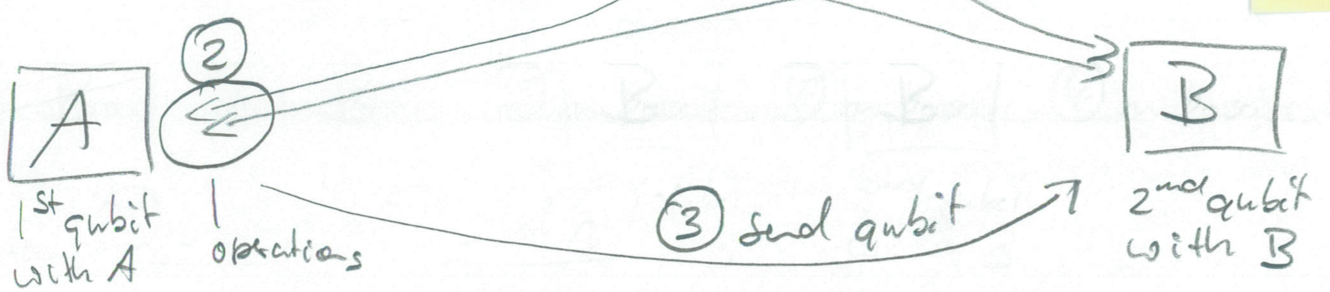
<u>Protocol:</u>

1. share entangled pair of qubits

1

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

original proposal by Wiesner and Bennett!




2. Alice performs one of 4 local operations on her bit

2 classical bits

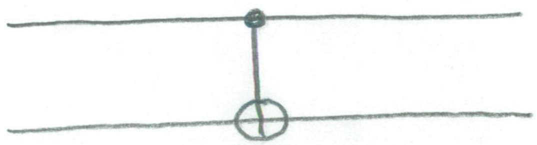
$$\left.\begin{matrix} I_1 \otimes I_2 \\ Z_1 \otimes I_2 \\ X_1 \otimes I_2 \\ iY_1 \otimes I_2 \end{matrix}\right\} \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) = \left\{\begin{matrix} \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) = \phi^+ & \longrightarrow & 0\,0 \\ \frac{1}{\sqrt{2}}\left(|00\rangle - |11\rangle\right) = \phi^- & \longrightarrow & 0\,1 \\ \frac{1}{\sqrt{2}}\left(|10\rangle + |01\rangle\right) = \psi^+ & & 0 \\ \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right) = \psi^- & & \end{matrix}\right.$$

What about physical realization?
→ slides: realization with photons!

3. Alice sends qubit to Bob
4. Bob performs a measurement on both qubits and finds 4 outcomes

# CNOT : A Universal 2-Qubit Logic Gate

Controlled NOT gate

$|A\rangle$  $|A\rangle$ CONTROL QUBIT

$|B\rangle$ $|A \oplus B\rangle$ TARGET QUBIT

> How would you realize a CNOT operation between two qubits?

truth table

| INPUT | OUTPUT |
|---|---|
| $\lvert 00\rangle$ | $\lvert 00\rangle$ |
| $\lvert 01\rangle$ | $\lvert 01\rangle$ |
| $\lvert 10\rangle$ | $\lvert 11\rangle$ |
| $\lvert 11\rangle$ | $\lvert 10\rangle$ |
| general | |
| $\lvert A, B\rangle$ | $\lvert A, A\oplus B\rangle$ |

addition mod 2

- is reversible (unitary)
- is universal
- can be realized using any two qubit interaction combined with single qubit manipulations

> What is required on a physical level to realize conditional logic?
> Do you know any types of interactions between quantum particles?

## Universality

Any multi qubit logic gate can be composed of CNOT and single qubit gates ($X, Y, Z, I$).

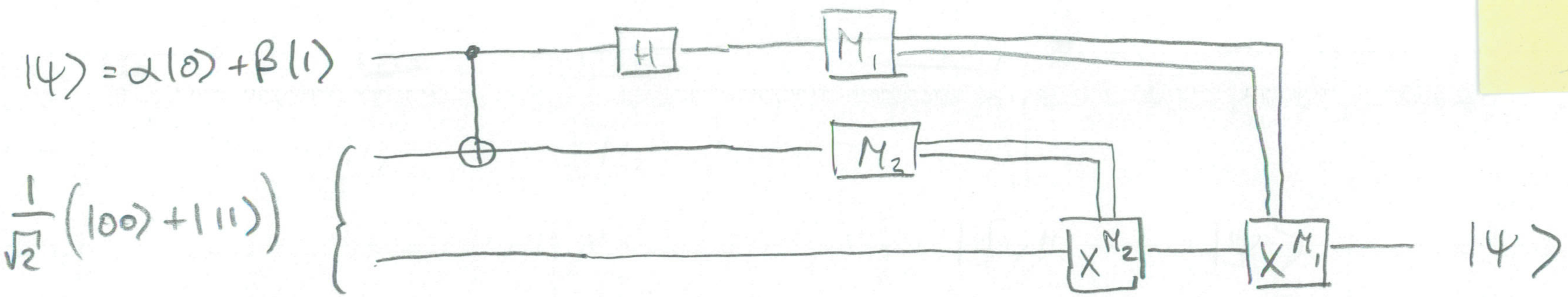
# Quantum Teleportation

**Task:** Transfer unknown quantum state $|\psi\rangle$ from Alice to Bob

**Resources:** entangled pair of qubits & classical communication

How would you perform this task?

**Circuit:**




INPUT (ALICE) OUTPUT (BOB)

**Steps:** (1) input (2) CNOT (3) Hadamard (4) measurement (5) conditional operations (6) output

**Note:**
- A has no information about $|\psi\rangle$ (and cannot obtain it)
- state is always fully transferred

# Teleportation Protocol

(1) Initial state $(\alpha|0\rangle + \beta|1\rangle)\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}(\alpha|000\rangle + \alpha|011\rangle + \beta|100\rangle + \beta|111\rangle)$

(2) $\xrightarrow{CNOT_{1,2}} \frac{1}{\sqrt{2}}(\alpha|000\rangle + \alpha|011\rangle + \beta|110\rangle + \beta|101\rangle)$

(3) $\xrightarrow{H_1}$
$$\frac{1}{2}\Big(\alpha|000\rangle + \alpha|100\rangle + \alpha|011\rangle + \alpha|111\rangle + \beta|010\rangle - \beta|110\rangle + \beta|001\rangle - \beta|101\rangle\Big)$$

What are the different measurement outcomes? With which prob. do they occur?

$$= \frac{1}{2}\Big(|00\rangle(\alpha|0\rangle + \beta|1\rangle) + |10\rangle(\alpha|0\rangle - \beta|1\rangle) + |01\rangle(\alpha|1\rangle + \beta|0\rangle) + |11\rangle(\alpha|1\rangle - \beta|0\rangle)\Big)$$

(4) measurement of qubit state

$M_1 \otimes M_2 \otimes I$

$P_{00} = P_{10} = P_{01} = P_{11} = \frac{1}{4}$

(5) conditional qubit manipulations on post measurement state $|\psi'\rangle$

$$\left.\begin{array}{l} |00\rangle : \hat{I}|\psi'\rangle \\ |10\rangle : \hat{Z}|\psi'\rangle \\ |01\rangle : \hat{X}|\psi'\rangle \\ |11\rangle : \hat{X}\hat{Z}|\psi'\rangle \end{array}\right\} = \alpha|0\rangle + \beta|1\rangle = |\psi\rangle$$

- requires transfer of two bits of classical information to <u>B</u>ob to perform local operations that recover the original state

**Note:** – state of one qubit transferred using one pair of entangled qubits and two bits of classical information

⇒ task cannot be performed classically

**Applications:**
- quantum error correction
- quantum gates
- quantum repeaters

original proposal: C. H. Bennett et al. Phys. Rev. Lett <u>70</u>, 1895 (1993)

first experimental implementation: D. Bouwmeester et al. Nature <u>390</u>, 575 (1997)

⇒ tested in different implementations using
- photons
- nuclear spins
- ions

⇒ hallmark quantum information processing experiment