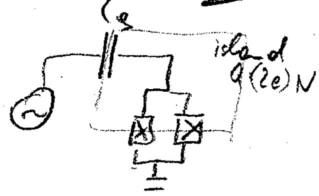
Jaynes Cummings for circuit QED:



1) Two-level approximation of CPB Hamiltonian

$$H = \sum_N \left\{ E_C\,(N-N_g)^2\,|N\rangle\langle N| - \frac{E_J}{2}\left(|N\rangle\langle N+1| + |N+1\rangle\langle N|\right)\right\}$$

$$E_C = \frac{(2e)^2}{2C_\Sigma} \qquad E_J = E_{J0}\cos\delta$$

take only $N = 0, 1$ into account:

$$H_2 = E_C N_g^2\,|0\rangle\langle 0| + E_C(1-N_g)^2\,|1\rangle\langle 1| - \frac{E_J}{2}\underbrace{\left(|0\rangle\langle 1| + |1\rangle\langle 0|\right)}_{\bar\sigma_x}$$

$$= -\frac{E_{el}}{2}|0\rangle\langle 0| + \frac{E_{el}}{2}|1\rangle\langle 1| - \frac{E_J}{2}\bar\sigma_x$$

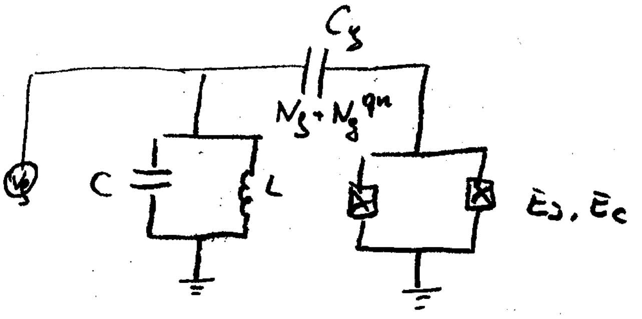
$\Rightarrow$ shift of energy: $E_{el} = E_C(1-2N_g)$

$$\Rightarrow\ = -\frac{E_{el}}{2}\bar\sigma_z - \frac{E_J}{2}\bar\sigma_x$$

$\Rightarrow$ Eigenbasis by rotation about y-axis: $e^{i\frac{\theta}{2}\sigma_y}$: $\sigma_x = \cos\theta\,\bar\sigma_x + \sin\theta\,\bar\sigma_z$, $\sigma_z = -\sin\theta\,\bar\sigma_x + \cos\theta\,\bar\sigma_z$

$$H_2 = \frac{1}{2}\begin{pmatrix} -E_{el} & -E_J \\ -E_J & E_{el} \end{pmatrix} \qquad \Rightarrow H_2 = \frac{\hbar\Omega}{2}\sigma_z \qquad \theta = \arctan\left(\frac{E_J}{E_{el}}\right)$$

2) Coupling to gate capacitor:



N_g .. polarization charge on C_g:

$$N_g = \frac{C_g V_g}{2e}$$

$$H = \frac{1}{2}E_C\left(1 - 2(N_g + \hat N_g^{qu})\right)\bar\sigma_z - \frac{E_J}{2}\bar\sigma_x$$

For simplicity $N_g = \frac{1}{2}$: $\quad H = \frac{E_C}{2}\left(\hat N_g^{qu}\,\bar\sigma_z - \frac{E_J}{2}\bar\sigma_x\right) =$

charge sweet spot

$$\hat Q^{qu} = (2e)\hat N_g^{qu} = C_g\hat V_g \qquad = \frac{E_C}{2}\underbrace{\frac{C_g}{2e}\sqrt{\frac{\hbar\omega}{2C}}\left(\hat a^+ + \hat a\right)}_{C_g \hat V_g / 2e}\bar\sigma_z - \frac{E_J}{2}\bar\sigma_x$$

$\theta = \arctan\left(\frac{E_J}{E_L}\right) \quad E_L \sim 0 \;@ N_g = \frac{1}{2} \quad \theta = \frac{\pi}{2}$ (6)

$\Rightarrow \quad \sigma_x = \bar{\sigma}_z \qquad \sigma_z = -\bar{\sigma}_x \qquad\qquad E_C = \frac{(2e)^2}{2C_\Sigma}$

$$= e \frac{C_g}{C_\Sigma}\sqrt{\frac{\hbar\omega_1}{2C}}\,\underbrace{(a^+ + a)\,\sigma_x}_{a^+\sigma^+ + a\sigma^+ + a^+\sigma^- + a\sigma^-} + \frac{E_J}{2}\sigma_z$$

$$\sigma_x = (\sigma^+ + \sigma^-)$$

$\sigma^+ = \sigma_x + i\sigma_y$

$\sigma^- = \sigma_x - i\sigma_y$

$\cancel{a^+\sigma^+} + a\sigma^+ + a^+\sigma^- + \cancel{a\sigma^-}$ energy conservation, RWA

$\Rightarrow$ full circuit Hamiltonian

$$\boxed{\hat{H} = \hbar\omega_1\left(a^+a + \frac{1}{2}\right) + \frac{E_J}{2}\sigma_z + \underbrace{\frac{C_g}{C_\Sigma}2e\sqrt{\frac{\hbar\omega_1}{2C}}}_{\hbar g}\left(a\sigma^+ + a^+\sigma^-\right)}$$

$\frac{2g}{\pi}$ = Vacuum Rabi Frequency

Rotating Wave approximation:

apply time dependent signal to CPB by changing N_g
(classical coherent signal)

$\hbar = 1$

$$H = -\frac{E_{el}(t)}{2}\sigma_z - \frac{E_J}{2}\sigma_x \qquad E_{el}(t) = E_C\left(1 - 2\left[N_g(0) + \eta\cdot\cos\omega t\right]\right)$$

$$= -\frac{E_{el}(0)}{2}\sigma_z - \frac{E_J}{2}\sigma_x + \overbrace{2E_C\eta}^{\varepsilon}\cos\omega t\,\sigma_z$$

in Eigenbasis at $t=0$ for $N_g(0) = \frac{1}{2}$:

$$H = \frac{\Omega}{2}\sigma_z + \varepsilon\cos\omega t\,\sigma_x = \vec{m}(t)\cdot\vec{\sigma} \qquad \vec{m}(t) = \begin{pmatrix}\varepsilon\cos\omega t\\ 0\\ \Omega/2\end{pmatrix}$$

Larmor Precession of unperturbed system: $|\psi(t)\rangle = e^{-i\frac{\Omega}{2}\sigma_z t}\,|\psi(0)\rangle$

Take the component of $\vec{m}(t)$ which rotates at approx. the Larmor frequency:

$$\vec{m}(t) = \frac{1}{2}\begin{pmatrix}\varepsilon\cos\omega t\\ \varepsilon\sin\omega t\\ \Omega/2\end{pmatrix} + \frac{1}{2}\begin{pmatrix}\varepsilon\cos\omega t\\ -\varepsilon\sin\omega t\\ \Omega/2\end{pmatrix}$$

$$H = \frac{\Omega}{2}\sigma_z + \frac{\varepsilon}{2}\overbrace{\left[\cos\omega t\,\sigma_x + \sin\omega t\,\sigma_y\right]}^{①^+} + \frac{\varepsilon}{2}\overbrace{\left[\cos\omega t\,\sigma_x - \sin\omega t\,\sigma_y\right]}^{①^-}$$

Rotating Frame $U = e^{i\frac{\tilde{\omega}t}{2}\sigma_z}$ $\qquad UHU^\dagger \underbrace{U|\psi\rangle}_{|\phi\rangle} = i\hbar\, U\frac{d}{dt}(U^\dagger\underbrace{U|\psi\rangle}_{|\phi\rangle}) = i\hbar\left(U\dot{U}^\dagger + \frac{d}{dt}|\phi\rangle\right)$

$$\tilde{H} = UHU^\dagger - i\,U\dot{U}^\dagger: \qquad U\sigma_z U^\dagger = \sigma_z$$

$$U\sigma_x U^\dagger = \begin{pmatrix}e^{\frac{i\tilde{\omega}t}{2}} & 0\\ 0 & e^{-\frac{i\tilde{\omega}t}{2}}\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}e^{\frac{i\tilde{\omega}t}{2}} & 0\\ 0 & e^{-\frac{i\tilde{\omega}t}{2}}\end{pmatrix}.$$

$$= \cos\tilde{\omega}t\,\sigma_x - \sin\tilde{\omega}t\,\sigma_y$$

$$U\sigma_y U^\dagger = \sin\tilde{\omega}t\,\sigma_x + \cos\tilde{\omega}t\,\sigma_y$$

$$U\dot{U}^\dagger = -i\frac{\tilde{\omega}}{2}\sigma_z$$

$①^\pm$: $\cos\omega t\left(\cos\tilde{\omega}t\,\sigma_x - \sin\tilde{\omega}t\,\sigma_y\right) \pm \sin\omega t\left(\sin\tilde{\omega}t\,\sigma_x + \cos\tilde{\omega}t\,\sigma_y\right) =$

$$\tilde{\omega} = \omega = \begin{cases} + : & (\cos^2\omega t + \sin^2\omega t)\sigma_x = \sigma_x \\ - : & (\cos 2\omega t\,\sigma_x - \sin 2\omega t\,\sigma_y) \text{ -- rotating at twice the frequency}\end{cases}$$

$\tilde{H} = \frac{1}{2}(\Omega - \omega)\sigma_z + \frac{\varepsilon}{2}\sigma_x$ $\Rightarrow$ transformation to static Hamiltonian by discarding ①$^-$, which is called the Rotating Wave Approx.

@ resonance for $\omega = \Omega$:

$\tilde{H}_{res} = \frac{\varepsilon}{2}\sigma_x \qquad \tilde{U} = e^{-\frac{i}{\hbar}\tilde{H}_{res}t}$ .. rotation about x-axis

Rotating Wave approximation in Jaynes - Cummings Model:

Rotating Frame Transformation: (Interaction Picture)

$$U = \underset{\text{qubit}}{e^{i\frac{\Omega}{2}t\sigma_z}} \otimes \underset{\text{resonator}}{e^{i\omega_r a^\dagger a}} = U_q \otimes U_r$$

$$U H_0 U^\dagger = U\left[\omega_r(a^\dagger a + \tfrac{1}{2}) + \tfrac{\Omega}{2}\sigma_z\right]U^\dagger =$$

$$= \omega_r\left[U_r a^\dagger a U_r^\dagger + \tfrac{1}{2}\right] + \tfrac{\Omega}{2}U_q\sigma_z U_q^\dagger = H_0$$

$$-iU\dot{U}^\dagger = -\tfrac{\Omega}{2}\sigma_z - \omega_r a^\dagger a$$

$$\to \tilde{H}_0 = 0$$

$$\hbar g\, U(a^\dagger + a)(\sigma^+ + \sigma^-)U^\dagger = (\underbrace{U_r a^\dagger U_r^\dagger}_{a^\dagger e^{i\omega_r t}} + \underbrace{U_r a U_r^\dagger}_{a e^{-i\omega_r t}})(\underbrace{U_q\sigma^+U_q^\dagger}_{\sigma^+ e^{i\Omega t}} + \underbrace{U_q\sigma^-U_q^\dagger}_{\sigma^- e^{-i\Omega t}})$$

$$\to a^\dagger\sigma^+ e^{i(\omega_r+\Omega)t} + a\sigma^- e^{-i(\omega_r+\Omega)t} +$$

$$+ a\sigma^+ + a^\dagger\sigma^-$$

only those are kept

Bloch equations:

$$H = -\mu\,\vec\sigma\cdot\vec m(t) \qquad \vec B(t) = \begin{pmatrix} m_x \\ m_y \\ m_z \end{pmatrix}$$

$$\dot\rho = -\frac{i}{\hbar}[H,\rho] \qquad \rho = |\psi\rangle\langle\psi| = \frac{1}{2}(1 + \vec r\cdot\vec\sigma)$$

$$\dot\rho = +\frac{i\mu}{\hbar}\left[\vec\sigma\cdot\vec m\cdot\frac{1}{2}(1+\vec r\vec\sigma) - \frac{1}{2}(1+\vec r\cdot\vec\sigma)\cdot\vec\sigma\vec m\right] =$$

$$= +\frac{i\mu}{2\hbar}\left[(\vec m\cdot\vec\sigma)(\vec r\cdot\vec\sigma) - (\vec r\cdot\vec\sigma)(\vec m\cdot\vec\sigma)\right]$$

using $(\vec a\cdot\vec\sigma)(\vec b\cdot\vec\sigma) = (\vec a\cdot\vec b)\mathbb{1} + i(\vec a\times\vec b)\cdot\vec\sigma$

$$= +\frac{i\mu}{\hbar}\left(-i\,\vec r\times\vec m\right)\vec\sigma$$

$$= \frac{\mu}{\hbar}(\vec r\times\vec m)\cdot\vec\sigma$$

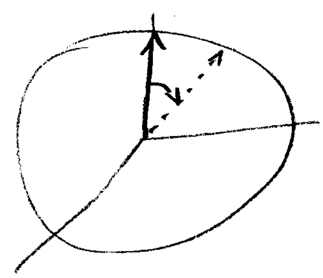
in terms of Bloch vector components:

$$\dot{\vec r} = \mathrm{Tr}(\dot\rho\,\vec\sigma) = \overset{\gamma}{\frac{\mu}{\hbar}}(\vec r\times\vec m) \quad \dots \text{ "Bloch equations"}$$

$m_y = 0$

$$\dot r_x = \gamma\, r_y m_z$$
$$\dot r_y = \gamma\,(r_z m_x - r_x m_z)$$
$$\dot r_z = -\gamma\, r_y m_x$$

e.g.: $\vec r(0) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow \dot r_y \neq 0$



Problem: no relaxation to thermal equilibrium

e.g. $\vec m = m_z$, system in excited state: $(|r_z| = 1) \Rightarrow \dot r = 0$

introduce longitudinal and transversal relaxation

$$\dot{r}_x = \gamma \, (\vec{r} \times m)_x - \frac{r_x}{T_2}$$

$$\dot{r}_y = \gamma \, (\vec{r} \times m)_y - \frac{r_y}{T_2}$$

$$\dot{r}_z = \gamma \, (\vec{r} \times m)_z - \frac{r_z - r_z^s}{T_1}$$

r_z^s ... steady state (-1 for ground state)

now: with $r_z^s = -1$: $\quad \dot{r}_z = -(r_z + 1) \cdot \frac{1}{T_1}$

$$\rightarrow \quad r_z(t) = e^{-\frac{t}{T_1}} (\cancel{r_z}\, 1 + r_z(0)) - 1$$

$\rightarrow$ T_1: longitudinal relaxation time

e.g.: $r_x(0) = 1 \qquad \vec{m} = m_z \, ; \quad T_1 = 0$

$$\left.\begin{aligned} \dot{r}_y &= -\gamma \, r_x m_z - \frac{r_y}{T_2} \\ \dot{r}_x &= \gamma \, r_y m_z - \frac{r_x}{T_2} \end{aligned}\right\} \; r_x(t), r_y(t) \propto e^{-\frac{1}{T_2}}$$

T_2: transverse relaxation time, dephasing

rotating frame rate $\omega = \gamma B_z$:
$$r_x(t) = \cos \omega t \; \tilde{r}_x(t) + \sin \omega t \; \tilde{r}_y(t)$$
$$r_y(t) = -\sin \omega t \; \tilde{r}_x(t) + \cos \omega t \; \tilde{r}_y(t)$$