Representation of a single qubit state on the Bloch sphere

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ with $\alpha\alpha^* + \beta\beta^* = 1$

$\alpha, \beta \in \mathbb{C}$

$\Rightarrow$ 4 parameters $\mathrm{Re}[\alpha]$, $\mathrm{Re}[\beta]$, $\mathrm{Im}[\alpha]$, $\mathrm{Im}[\beta]$

+1 normalization constraint

rewrite:

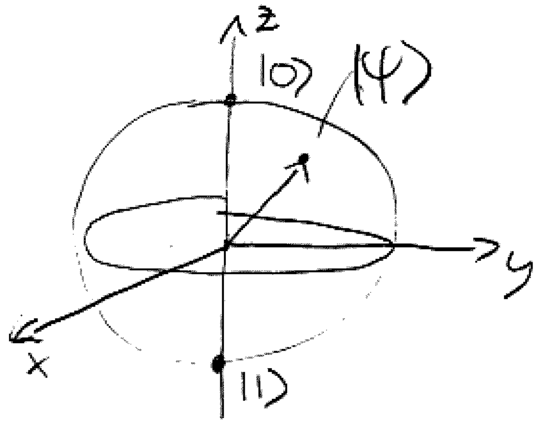
$$|\psi\rangle = e^{i\gamma}\left[\cos \theta/2\, |0\rangle + e^{i\varphi} \sin \theta/2\, |1\rangle\right]$$

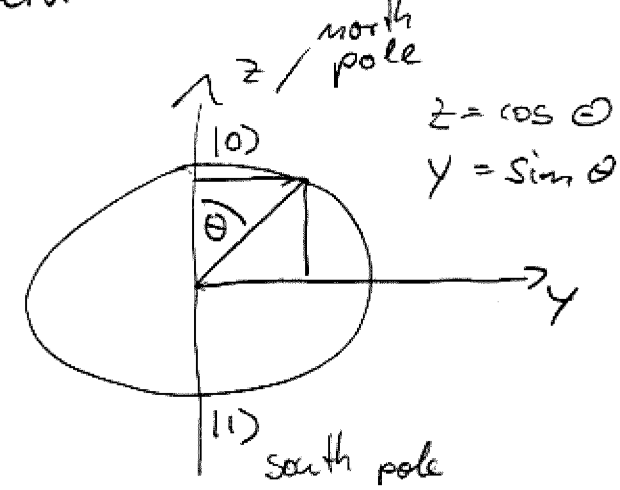
vector pointing to the surface of a unit sphere

θ : polar angle

φ : azimuth angle

γ : global phase factor

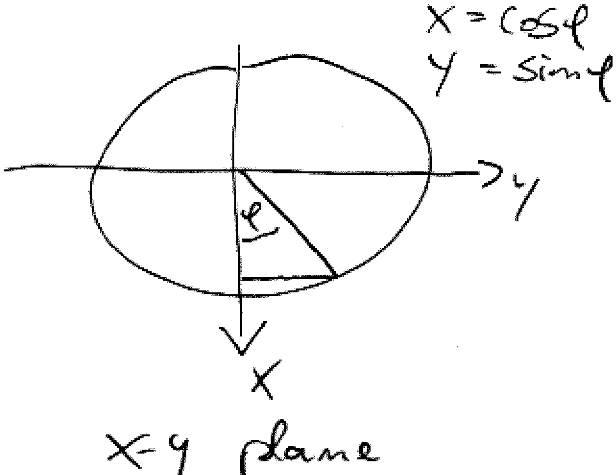




$z = \cos\theta$

$y = \sin\theta$

y - z plane



$x = \cos\varphi$

$y = \sin\varphi$

x-y plane

$|\psi\rangle = (\cos\varphi \sin\theta, \sin\varphi \sin\theta, \cos\theta)$

$|0\rangle = (0,0,1)$; $\theta = 0$

$|1\rangle = (0,0,-1)$; $\theta = \pi$

basis states

Superposition states with equal probabilits

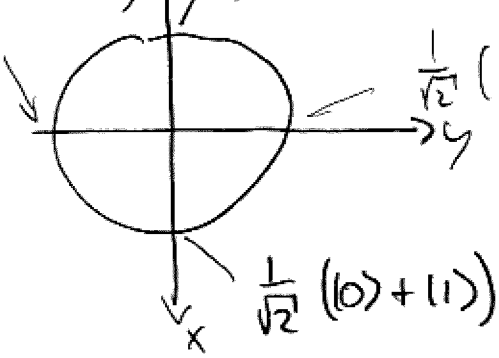
$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + \underset{\underset{\cos\varphi + i\sin\varphi}{\|}}{e^{i\varphi}}|1\rangle\right) \Rightarrow \Theta = \frac{\pi}{2}$$

$$\cos\frac{\pi}{4} = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$\varphi = \frac{3}{2}\pi$

$\frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$

$\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) : \varphi = \pi$



$\frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle) : \varphi = \frac{\pi}{2}$

$\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) : \varphi = 0$

x-y plane of Bloch sphere

Single qubit logic operations

$$|\psi\rangle_{in} \;\frac{\qquad\quad}{}\; \boxed{\hat{O}} \;\frac{\qquad\quad}{}\; |\psi\rangle_{out}$$

$\hat{O}$: $\hat{X}, \hat{Y}, \hat{Z}, \hat{I}$ Pauli matrices

$$\hat{X} = \hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\hat{Y} = \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\hat{Z} = \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\hat{I} = \hat{\mathbb{1}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Action of operations

$$\hat{X}|0\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle$$

$$\hat{X}|1\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle$$

bit flip

$$\hat{Y}|0\rangle = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = i\begin{pmatrix} 0 \\ 1 \end{pmatrix} = i\,|1\rangle$$

$$\hat{Y}|1\rangle = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -i\begin{pmatrix} 1 \\ 0 \end{pmatrix} = -i\,|0\rangle$$

conjugate bit flip

$$\hat{Z}|0\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle$$

$$\hat{Z}|1\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -|1\rangle$$

phase flip (by π)

$\Rightarrow$ interpret operation on Bloch sphere

$$\hat{I}\,|0\rangle = |0\rangle$$
$$\hat{I}\,|1\rangle = |1\rangle$$

<u>identity operation</u>

- repeated action of Pauli matrices

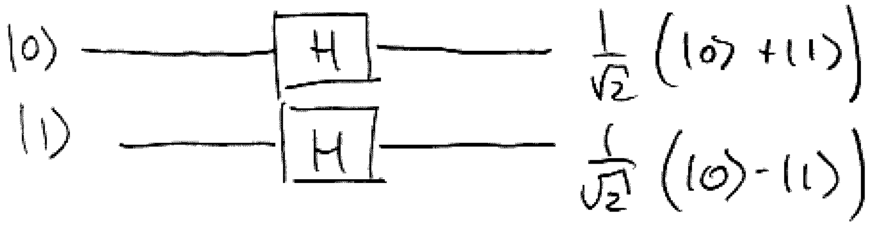
$$\hat{X}\hat{X} = \begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix} = \begin{pmatrix}1&0\\0&1\end{pmatrix} = \hat{I}$$

$$\left.\begin{array}{ll} \hat{X}^{2m} = \hat{I} & m = 1,2,3,\ldots \\ \hat{X}^{2m+1} = \hat{X} & m = 1,2,3,\ldots \end{array}\right\}$$

interpret in terms of repeated flips on Bloch sphere

equivalently for $\hat{Y}, \hat{Z}$

- The Hadamard gate



$$H = \frac{1}{\sqrt{2}}(X+Z) = \frac{1}{\sqrt{2}}\left(\begin{pmatrix}0&1\\1&0\end{pmatrix} + \begin{pmatrix}1&0\\0&-1\end{pmatrix}\right)$$
$$= \frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}$$

$$\hat{H}\hat{H} = \frac{1}{2}\begin{pmatrix}1&1\\1&-1\end{pmatrix}\begin{pmatrix}1&1\\1&-1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}2&0\\0&2\end{pmatrix} = \begin{pmatrix}1&0\\0&1\end{pmatrix} = \hat{I}$$

Dynamics of a single qubit

- consider spin 1/2 particle in an external magnetic field $B = (0,0,B_z)$

- Hamiltonian $H = -\vec{\mu}\cdot\vec{B}$

g-factor

$\frac{e\hbar}{2m}$ Bohr magneton

- Hamilton Operator

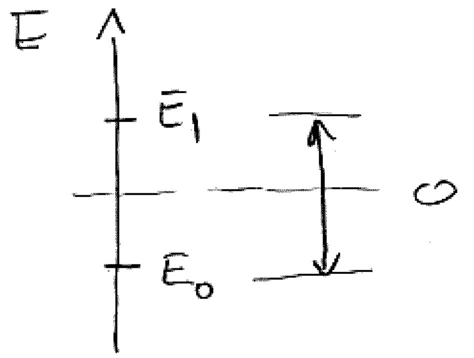
$$\hat{H} = -g\mu_B B_z \frac{1}{2} \hat{Z}$$

- Eigenvalues and Eigenvectors { solutions to the time-independent Schrödinger equation

$$\hat{H}|0\rangle = -\frac{g\mu_B B_z}{2}|0\rangle = E_0|0\rangle$$

$$\hat{H}|1\rangle = \frac{g\mu_B B_z}{2}|1\rangle = E_1|1\rangle$$

- energy level diagram



$$E_{1,0} = \pm\frac{g\mu_B B_z}{2}$$

$$\equiv \pm\frac{\hbar\Omega_z}{2}$$

- energy level separation

$$\Delta E = E_1 - E_0$$

$$= \hbar\Omega_z = g\mu_B B_z$$

- time evolution described by the time dependent Schrödinger equation

$$i\hbar\frac{d}{dt}|\psi\rangle = \hat{H}|\psi\rangle$$

=> solution by separation of variables

- general solution for <u>time independent</u> Hamiltonian

↓ bias fields do not change in time

$$|\psi(t)\rangle = \exp\left(-\frac{i}{\hbar}\hat{H}t\right)|\psi(0)\rangle$$

- Remark on operator exponentiation

$$\exp\left(i\Theta\hat{O}\right) = \cos\Theta\,\hat{I} + i\sin\theta\,\hat{O}$$

for $\Theta \in \mathbb{R}$ and $\hat{O}^2 \underset{\uparrow}{=} \hat{I}$

true for Pauli matrices

$$\exp X = \sum^{\infty} \frac{X^n}{n!}$$ Taylor expansion

$$\exp\left(i\Theta\hat{O}\right) = \sum_{n=0}^{\infty} \frac{\left(i\Theta\hat{O}\right)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{\left(i\Theta\hat{O}\right)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{\left(i\Theta\hat{O}\right)^{2n+1}}{(2n+1)!}$$

$$= \cos\Theta\,\hat{I} + i\sin\theta\,\hat{O}$$

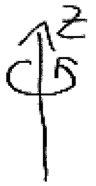
- <u>example</u>: $\hat{H} = -\frac{\hbar\Omega_z}{2}\hat{Z}$ ⟸ Hamiltonian of electron spin in magnetic field.

$$\exp\left(-\frac{i}{\hbar}\left(-\frac{\hbar\Omega_z}{2}\right)t\,\hat{Z}\right) = \exp\left(i\frac{\Omega_z t}{2}\hat{Z}\right)$$

$$= \cos\Theta_z/2\,\hat{I} + i\sin\Theta_z/2\,\hat{Z} = \underset{\Uparrow}{R_z}(\Theta_z)$$

rotation operator

with $\Theta_z = \Omega_z t$



on Bloch sphere

Example: Dynamics of a superposition state

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle \right)$$ the initial state

$$\hat{H} = -\frac{\hbar \Omega_z}{2} \hat{Z} = -\frac{\hbar\Omega_z}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = -\frac{\hbar\Omega_z}{2}\left(|0\rangle\langle 0| - |1\rangle\langle 1| \right)$$

the Hamiltonian

$$|\psi(t)\rangle = \exp\left(+ \frac{i}{\hbar} \frac{\hbar \Omega_z}{2} t \left(|0\rangle\langle 0| - |1\rangle\langle 1| \right)\right) |\psi(0)\rangle$$

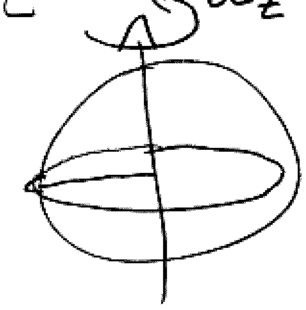
$$= \left(\exp i \frac{\Omega_z t}{2} |0\rangle + \exp -i \frac{\Omega_z t}{2} |1\rangle \right) \frac{1}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}}\left(|0\rangle + e^{-i\Omega_z t} |1\rangle \right)$$

vgl. mit

$$e^{i\gamma}\left(\cos\frac{\theta}{2} |0\rangle + i \sin\frac{\theta}{2} e^{+i\varphi} |1\rangle \right)$$

$$\Rightarrow \theta = \frac{\pi}{2} \ ; \ \varphi = -\Omega_z t$$



$\downarrow$

rotation of qubit state vector around z-axis

$\Rightarrow$ $\boxed{\text{Larmor precession}}$ at Larmor frequency Ω_z

Measurement of a qubit in the computational basis

- the measurement operators

$$\hat{M}_0 = |0\rangle\langle 0| = \begin{pmatrix}1\\0\end{pmatrix}(1\,0) = \begin{pmatrix}1&0\\0&0\end{pmatrix}$$

$$\hat{M}_1 = |1\rangle\langle 1| = \begin{pmatrix}0\\1\end{pmatrix}(0\,1) = \begin{pmatrix}0&0\\0&1\end{pmatrix}$$

- completeness

$$\sum_i \hat{M}_i^+ \hat{M}_i = \begin{pmatrix}1&0\\0&0\end{pmatrix} + \begin{pmatrix}0&0\\0&1\end{pmatrix} = \hat{I}$$

- measurement of state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

$$P_0 = \langle\psi| M_0^+ M_0 |\psi\rangle$$

$$= \left(\alpha^*\langle 0| + \beta^*\langle 1|\right)\left(|0\rangle\langle 0|\right)^+ |0\rangle\langle 0| \left(\alpha|0\rangle + \beta|1\rangle\right)$$

$$= \left(\alpha^*\langle 0| + \beta^*\langle 1|\right) |0\rangle\langle 0| \left(\alpha|0\rangle + \beta|1\rangle\right)$$

$$= \alpha^*\alpha = |\alpha|^2$$

$$P_1 = \langle\psi| M_1^+ M_1 |\psi\rangle = \beta^*\beta = |\beta|^2$$

Note:
- a single preparation of a state followed by a single measurement results in a single measurement outcome, 0 or 1 with prob. $|\alpha|^2$ or $|\beta|^2$
- to find probabilities $|\alpha|^2$, $|\beta|^2$ the state has to be prepared and measured repeatedly
- even then you find only the probabilities $|\alpha|^2$ and $|\beta|^2$ not $\alpha, \beta \in \mathbb{C}$
- measure not only z-projections but also x, y projections to fully recover state ⇝ Tomography

- Post measurement state

$$|\psi_0\rangle = \frac{M_0|\psi\rangle}{\sqrt{P_0}} \cdot \frac{|0\rangle\langle 0|(\alpha|0\rangle+\beta|1\rangle)}{\sqrt{|\alpha|^2}} = \frac{\alpha}{|\alpha|}|0\rangle$$

$$|\psi_1\rangle = \frac{M_1|\psi\rangle}{\sqrt{P_1}} = \frac{|1\rangle\langle 1|(\alpha|0\rangle+\beta|1\rangle)}{\sqrt{\beta^2}} = \frac{\beta}{|\beta|}|1\rangle$$

- repeated measurement

$$P_{00} = \langle\psi_0|M_0^+M_0|\psi_0\rangle = \frac{\beta^*}{|\beta|}\langle 0||0\rangle\langle 0||0\rangle\langle 0|0\rangle\frac{\beta}{|}$$

$$= 1$$

$P_{01} = 0$

$P_{11} = 1$

$P_{10} = 0$

↑ checks for outcome of repeated measurement

⇒ this property is to be checked for any realization of a projective measurement

Potential issues:
- qubit spontaneous emission
- stimulated emission or absorbtion due to measurement apparatus
- misidentification of state by measurement apparatus

Product or separable two qubit states:

$$|\psi_0\rangle = \alpha_0|0\rangle + \beta_0|1\rangle$$

$$|\psi_1\rangle = \alpha_1|0\rangle + \beta_1|1\rangle$$

$$|\psi_0\rangle|\psi_1\rangle = |\psi_0\psi_1\rangle = \alpha_0\alpha_1|00\rangle + \alpha_0\beta_1|01\rangle + \alpha_1\beta_0|10\rangle + \alpha_1\beta_1|11\rangle$$

- two qubit state that can be factored into single qubit states

Is the following 2-qubit state separable?

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

$$\Rightarrow \quad \alpha_0\alpha_1 = \beta_0\beta_1 = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \alpha_0, \alpha_1, \beta_0, \beta_1 \neq 0$$

otherwise the product could not be different from 0

therefore

$$\left.\begin{aligned} \alpha_0\beta_1 &\neq 0 \\ \alpha_1\beta_0 &\neq 0 \end{aligned}\right\}$$ state can not be written as a product state!

⇒ This is an <u>entangled</u> state

Measurement of an entangled state

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

- probabilits to find qubit 1 in state $|0\rangle$

$$P_0^{(1)} = \frac{1}{2}\left(\langle 00| + \langle 11|\right)\left(M_0 \otimes I\right)^{+}\left(M_0 \otimes I\right)\left(|00\rangle + |11\rangle\right)$$

$$= \frac{1}{2}\left(\underbrace{\langle 0|M_0^{+}}_{\langle 0|} \otimes \langle 0|I + \cancel{\langle 1|M_0^{+} \otimes \langle 1|I}\right)\left(\underbrace{M_0|0\rangle}_{|0\rangle} \otimes I|0\rangle + \cancel{M_0|1\rangle \otimes I|1\rangle}\right.$$

$$= \frac{1}{2}\langle 00|00\rangle = \frac{1}{2}$$

$P_1^{(1)} = \frac{1}{2}$ probabilits to find qubit 1 in state $|1\rangle$

- post measurement state

$$|\psi_0^{(1)}\rangle = \frac{\left(M_0 \otimes I\right)|\psi\rangle}{\sqrt{\frac{1}{2}}} = \frac{\frac{1}{\sqrt{2}}\left(\overbrace{M_0|0\rangle}^{|0\rangle} \otimes \overbrace{I|0\rangle}^{|0\rangle} + M_0|1\rangle \otimes I|1\rangle\right)}{1/\sqrt{2}}$$

$$= |00\rangle$$

$P_0^{(2)} = 1$ ⇐ projects second qubit with certaints into corresponding state

$P_1^{(2)} = 0$

⟹ the measurement outcomes are correlated

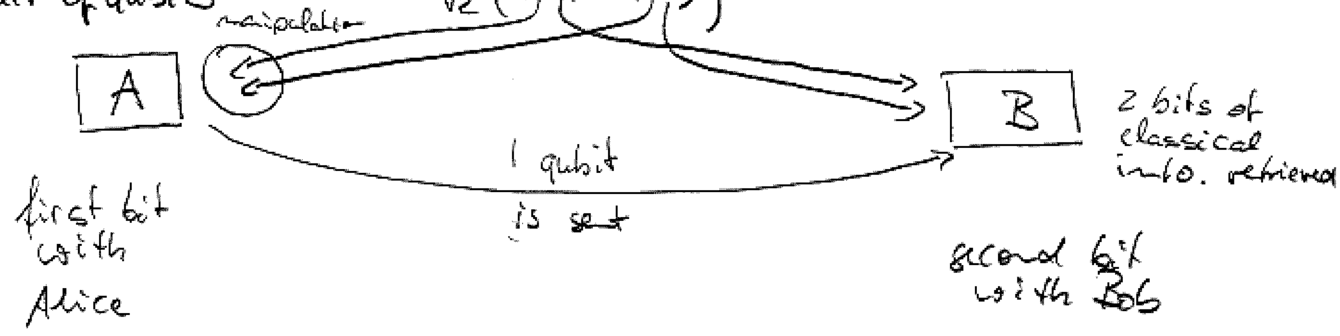
⟹ these correlations are stronger than in any classical systems

⇒ test of correlations in Bell inequalities

Super Dense Coding:

① share entangled pair of qubits

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$



② local operations by Alice

symmetric

2 bits of classica informati

$$(I_1 \otimes I_2)\frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) : \phi^+ : 00$$

$$(Z_1 \otimes I_2)\frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) = \frac{1}{\sqrt{2}}\left(|00\rangle - |11\rangle\right) : \phi^- : 01$$

$$(X_1 \otimes I_2)\frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) = \frac{1}{\sqrt{2}}\left(|10\rangle + |01\rangle\right) : \psi^+ : 10$$

$$(iY_1 \otimes I_2)\frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) = \boxed{\frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right)} : \psi^- : 11$$

$$i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

anti-symmetric

③ Alice sends her qubit to Bob

④ Bob performs joint measurements on both qubits and finds four different measurement outcomes.

Note: – two qubits are involved in protocol

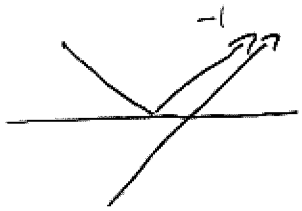
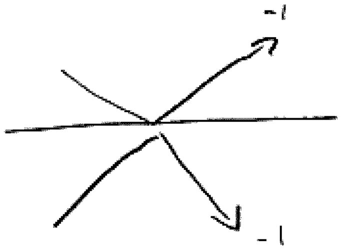
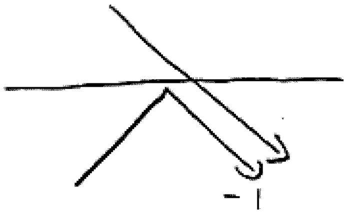
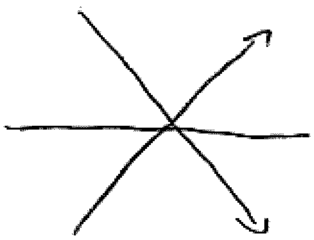
– However, Alice only interacts with one qubit and sends only one qubit along the channel

– Nevertheless two bits of classical information are transmitted!

$\Rightarrow$ special!

Action of a polarization independent beam splitter

②

		symmetric spatial w.f. "bosonic" → bunch	anti-symmetric spatial part "fermionic" → anti-bunch
①		X	
②			X
③		X	
④			X

$C_{HV} = 1$ $\qquad \psi^+ = \frac{1}{\sqrt{2}}\left(|HV\rangle + |VH\rangle\right)$ $\qquad \psi^- = \frac{1}{\sqrt{2}}\left(|HV\rangle - |VH\rangle\right)$

$$\Downarrow$$

$C_{HV'} = 1$

$C_{H\bar{H}} = 1$

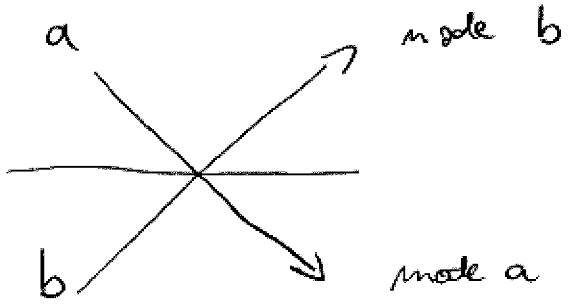
$$\left\{\begin{aligned} \phi^+ &= \frac{1}{\sqrt{2}}\left(|HH\rangle + |VV\rangle\right) \\ \phi^- &= \frac{1}{\sqrt{2}}\left(|HH\rangle - |VV\rangle\right) \end{aligned}\right.$$

indistinguishable as two photons go to the same detector

→ click only in one part of t4

symmetric polarization w.f.

anti-symmetric polarization w.f.



- polarization independent beam splitter

π phase shift upon reflection

π phase shift upon reflection

$$a_H^\dagger \xrightarrow{BS} \frac{1}{\sqrt{2}}\left(a_H^\dagger + i b_H^\dagger\right)$$

$$b_V^\dagger \xrightarrow{BS} \frac{1}{\sqrt{2}}\left(b_V^\dagger + i a_V^\dagger\right)$$

1 photon on beam splitter

$$\frac{1}{\sqrt{2}}\left(a_H^\dagger b_V^\dagger \overset{\text{antisym.}}{\underset{\text{sym.}}{\mp}} a_V^\dagger b_H^\dagger\right) \xrightarrow{BS}$$

two photons on beam splitter

$$\frac{1}{2}\frac{1}{\sqrt{2}}\left(\left(a_H^\dagger + i b_H^\dagger\right)\otimes\left(b_V^\dagger + i a_V^\dagger\right) \mp \left(a_V^\dagger + i b_V^\dagger\right)\otimes\left(b_H^\dagger + i a_H^\dagger\right)\right)$$

$$= \frac{1}{2\sqrt{2}}\Big(a_H^\dagger b_V^\dagger + i b_H^\dagger b_V^\dagger + i a_H^\dagger a_V^\dagger - b_H^\dagger a_V^\dagger$$
$$\pm b_V^\dagger a_H^\dagger \mp i b_H^\dagger b_V^\dagger \mp i a_V^\dagger b_H^\dagger \mp a_V^\dagger b_H\Big)$$

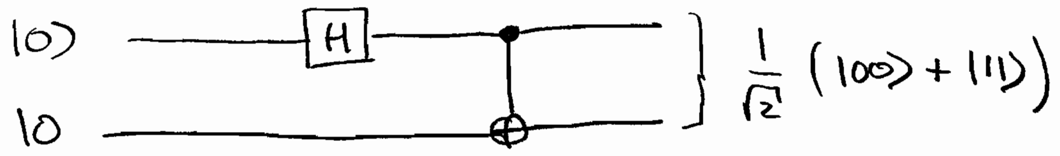
for − : $\frac{1}{\sqrt{2}}\left(a_H^\dagger b_V^\dagger - a_V^\dagger b_H^\dagger\right)$

for photons with anti-symmetric spatial w.f.
=> anti-bunch

for + : $i\frac{1}{\sqrt{2}}\left(a_H^\dagger a_V^\dagger + b_V^\dagger b_H^\dagger\right)$

for photons with symmetric spatial w.f.
=> bunch

Generation of entangled states (Bell states)



initial state $|00\rangle$

$$|00\rangle \xrightarrow{H_1} \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\,|0\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |10\rangle\right)$$

$$\xrightarrow{CNOT_{1,2}} \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) = \phi^+$$

$$|01\rangle \xrightarrow{H_1} \frac{1}{\sqrt{2}}\left(|01\rangle + |11\rangle\right)$$

$$\xrightarrow{CNOT_{1,2}} \frac{1}{\sqrt{2}}\left(|01\rangle + |10\rangle\right) = \psi^+$$

$$|10\rangle \xrightarrow{H_1} \frac{1}{\sqrt{2}}\left(|00\rangle - |10\rangle\right)$$

$$\xrightarrow{CNOT_{1,2}} \frac{1}{\sqrt{2}}\left(|00\rangle - |11\rangle\right) = \phi^-$$

$$|11\rangle \xrightarrow{H_1} \frac{1}{\sqrt{2}}\left(|01\rangle - |11\rangle\right)$$

$$\xrightarrow{CNOT_{12}} \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right) = \psi^-$$

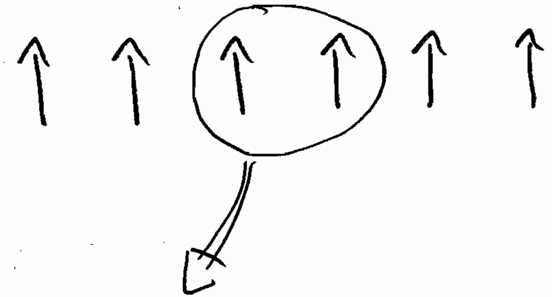
⇒ We will demonstrate how to generate these Bell states in various physical quantum systems

Implementation of a CNOT gate using a physical interaction

- need some physical two-qubit interaction to realize a CNOT gate
- in general a physical interaction does not automatically realize the CNOT gate
  - additional one qubit manipulations are needed

Example: Ising interaction

- consider chain of spins



$$H = \sum_{ij} -J_{ij} \hat{Z}_i \hat{Z}_j$$

- pairwise coupling between spins i and j
- no external magnetic field

- consider just two spins, i.e. two qubits

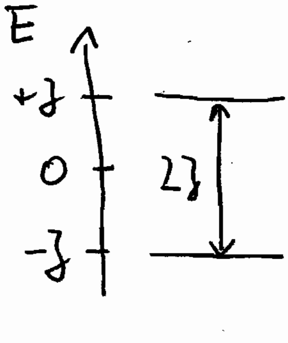
$$H = -J \underbrace{\hat{Z}_1}_{\pm 1} \underbrace{\hat{Z}_2}_{\pm 1} = -J \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Eigenvalues

- two cases:

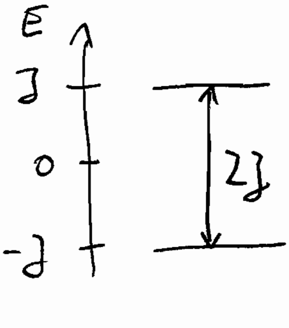
$J>0$: ferromagnetic coupling

$J<0$: anti-ferromagnetic coupling



$+1,-1$ ↑↓
$-1,+1$ ↓↑ $\}$ $-1\,(-J)$

$+1,+1$ ↑↑
$-1,-1$ ↓↓ $\}$ $+1\,(-J)$



$+1\,+1$ ↑↑
$-1\,-1$ ↓↓ $\}$ $+1\,(-(-J))$

$+1\,-1$ ↑↓
$-1\,+1$ ↓↑ $\}$ $-1\,(-(-J))$

– action of the $\hat{Z}\otimes\hat{Z}$ operator

qubits in groundstate

$$\hat{Z}\hat{Z}|00\rangle = (\hat{Z}\otimes\hat{Z})(|0\rangle\otimes|0\rangle)$$
$$= \hat{Z}|0\rangle\otimes\hat{Z}|0\rangle = |00\rangle$$

qubits in excited state

$$\hat{Z}\hat{Z}|11\rangle = \hat{Z}|1\rangle\otimes\hat{Z}|1\rangle = \underline{-|1\rangle\otimes-|1\rangle} = |11\rangle$$

adds phase factor to excited qubit state

qubits in superposition state

$$\hat{Z}\hat{Z}\left(\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)\otimes\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)\right)$$
$$= \frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)\otimes\frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)$$
$$= \frac{1}{2}\Big(\underbrace{\underbrace{|00\rangle+|11\rangle}_{\phi^+} - (\underbrace{|01\rangle+|10\rangle}_{\psi^+})}_{\text{superposition of two entangled states (Bell states)}}\Big)$$

⇒ introduces qubit state dependent phase factors

⇒ interaction generates [illegible] ( )

⇒ apply $\hat{Z}\hat{Z}$ again brings you back to the initial state

⇒ as in the case of one qubit controls using Pauli matrices

– unitary evolution under the Ising Hamiltonian

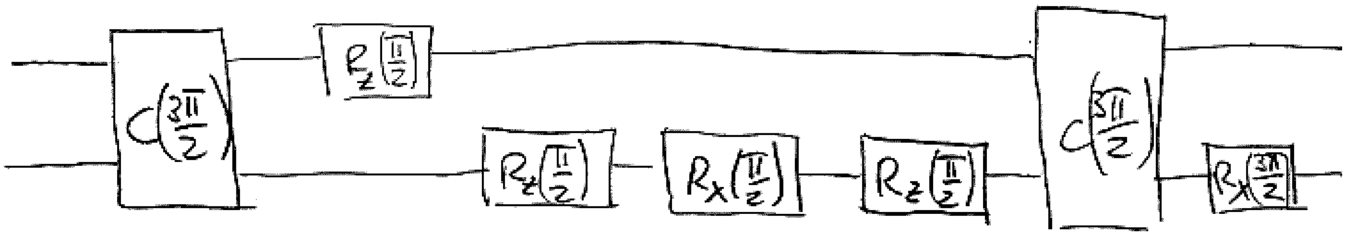
$$\exp\left(-\frac{i}{\hbar}(-J)\hat{Z}\hat{Z}\,t\right) = \cos\frac{J}{2\hbar}t\,\hat{I}\hat{I} - i\sin\frac{J}{2\hbar}t\,\hat{Z}\hat{Z}$$
$$= \cos\frac{\gamma}{2}\,\hat{I}\hat{I} - i\sin\frac{\gamma}{2}\,\hat{Z}\hat{Z}$$
$$= C(\gamma)$$

$\gamma = \frac{J}{\hbar}t$ 2-qubit phase rotation angle

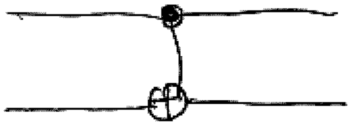
⇑

control J in time to generate two-qubit entangled states

Turning the Ising interaction into a CNOT gate



$\equiv$



phase factor

$$C_{NOT} = e^{-i\frac{3\pi}{4}} R_{x_2}\left(\frac{3\pi}{2}\right) C\left(\frac{3\pi}{2}\right) R_{z_2}\left(\frac{\pi}{2}\right) R_{x_2}\left(\frac{\pi}{2}\right) R_{z_2}\left(\frac{\pi}{2}\right) R_{z_1}\left(\frac{\pi}{2}\right) C\left(\frac{3\pi}{2}\right)$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \; !$$

$\Rightarrow$ try to work this out explicitly!

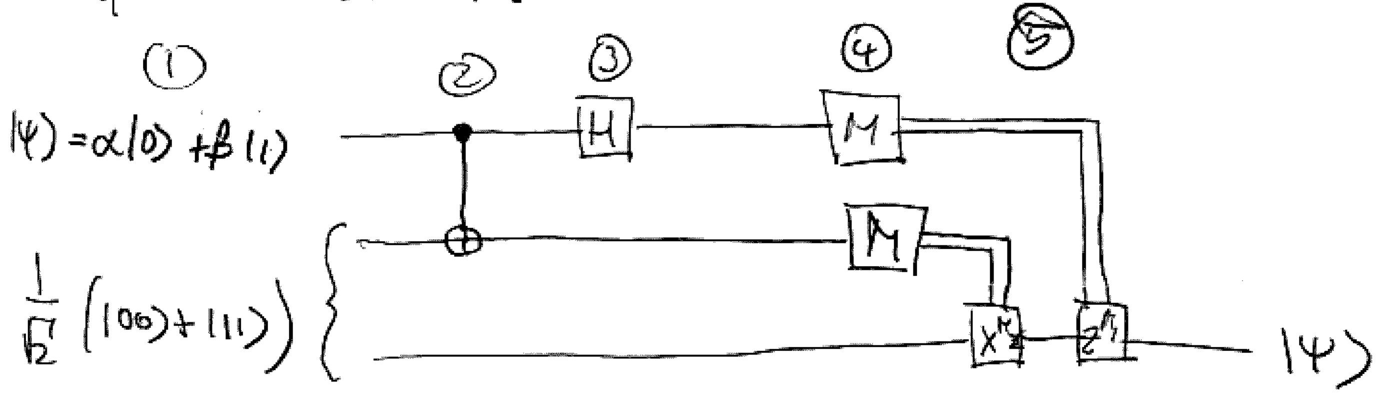
note:
- 2 two-qubit operations & 5 one-qubit operations
- useful to find optimal way to transform logical operations to physical interactions
- in this scheme one needs to switch on and off two-qubit interactions
  - $\Rightarrow$ how to realize that for the Ising interaction?
  - $\Rightarrow$ e.g. by controlling the detuning

Quantum Teleportation

- **Task:** transfer unknown quantum state $|\psi\rangle$ from A (Alice) to B (Bob) using an entangled pair and classical communication as a resource

- interesting aspects:
  - A does not have any information about the state
  - even in principle a could not gain full information about the state
  - nevertheless it can be fully transfered to B

- quantum circuit:



$\Rightarrow$ $|\psi\rangle$ can always be fully transfered from A to B

useful for:
- quantum error correction
- quantum gates

① initial state

$$(\alpha|0\rangle + \beta|1\rangle)\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \longrightarrow \frac{1}{\sqrt{2}}(\alpha|000\rangle + \alpha|011\rangle + \beta|100\rangle + \beta|111$$

② $$\xrightarrow{CNOT_{1,2}} \frac{1}{\sqrt{2}}\left(\alpha|000\rangle + \alpha|011\rangle + \beta|110\rangle + \beta|101\rangle\right)$$

③ $$\xrightarrow{H_1} \frac{1}{2}\Big(\alpha|000\rangle + \alpha|100\rangle + \alpha|011\rangle + \alpha|111\rangle$$
$$+\beta|010\rangle - \beta|110\rangle + \beta|001\rangle - \beta|101\rangle\Big)$$

$$= \frac{1}{2}\Big[|00\rangle(\alpha|0\rangle + \beta|1\rangle) \qquad \Rightarrow P_{00} = \frac{1}{4}$$
$$+|10\rangle(\alpha|0\rangle - \beta|1\rangle) \qquad \Rightarrow P_{10} = \frac{1}{4}$$
$$+|01\rangle(\alpha|1\rangle + \beta|0\rangle) \qquad \Rightarrow P_{01} = \frac{1}{4}$$
$$+|11\rangle(\alpha|1\rangle - \beta|0\rangle)\Big] \qquad \Rightarrow P_{11} = \frac{1}{4}$$

④ Measurement $M_1 \otimes M_2 \otimes I$

⑤ Measurement outcome dependent manipulation of qubit state

00 : $\psi_3 = \alpha|0\rangle + \beta|1\rangle$

10 : $\hat{Z}\psi_3 = \alpha|0\rangle + \beta|1\rangle$ phase flip

01 : $\hat{X}\psi_3 = \alpha|0\rangle + \beta|1\rangle$ bit flip

11 : $\underbrace{\hat{X}\hat{Z}\psi_3} = \alpha|0\rangle + \beta|1\rangle$ phase & bit flip

single qubit
rotations required to
recover the state

⇒ transfer of one qubit using one entangled pair and two bits of classical information

Electronic Harmonic Oscillator

20.10.2008 ①

- voltage across oscillator

$$V = \frac{Q}{C} = -L\dot{I} = -\dot{\Phi}$$

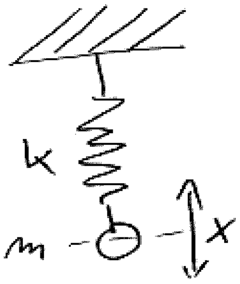
- Hamiltonian (total energy)

$$H = \underbrace{\frac{1}{2}CV^2}_{\text{electrostatic energy}} + \underbrace{\frac{1}{2}LI^2}_{\text{magnetic energy}} = \frac{Q^2}{2C} + \frac{\Phi^2}{2L}$$

- compare to mechanical harmonic oscillator

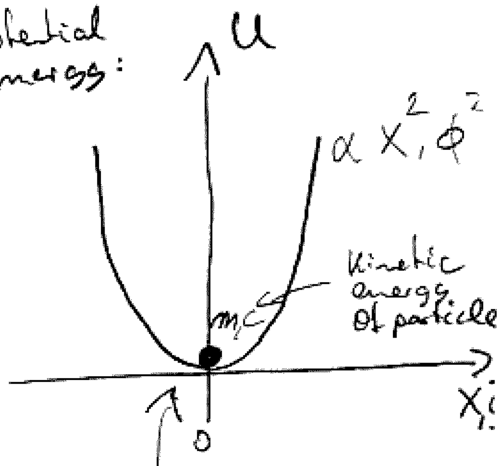
$$H = \underbrace{\frac{p^2}{2m}}_{\text{kinetic energy}} + \underbrace{\frac{1}{2}kx^2}_{\text{potential energy}} = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$$

with $\omega = \sqrt{\frac{k}{m}}$



mechanical	electronic
position X	flux Φ
momentum p	charge Q
mass m	Capacitance C
spring constant k	inverse inductance $\frac{1}{L}$
resonance frequency $\omega = \sqrt{\frac{k}{m}}$	resonance frequency $\omega = \frac{1}{\sqrt{LC}}$
conjugate variables x, p	conjugate variables Φ, Q

potential energy:



Curvature:

$$\frac{\partial^2 U}{\partial x^2} = k$$

$$\frac{\partial^2 U}{\partial \Phi^2} = \frac{1}{L}$$

Conjugate variables of the electronic oscillator

$$\frac{\partial H}{\partial \Phi} = \frac{\Phi}{L} = I = \dot{Q}$$

$$\frac{\partial H}{\partial Q} = \frac{Q}{C} = V = -L\dot{I} = -\dot{\Phi}$$

quantum commutation relations for the electronic harmonic oscillator

$$[\hat{\Phi}, \hat{Q}] = i\hbar$$ flux - charge

$$\left[2\pi \frac{\hat{\Phi}}{\Phi_0}, \frac{\hat{Q}}{2e}\right] = [\hat{\delta}, \hat{N}] = i$$ phase - number

Φ_0

$\shortparallel$

$\frac{h}{2e}$ magnetic flux quantum

compare with mechanical harmonic oscillator

$$[\hat{x}, \hat{p}] = i\hbar$$

$\hat{x} = x$

$\hat{p} = -i\hbar \frac{\partial}{\partial x}$

$$[\hat{\Phi}, \hat{Q}] = i\hbar$$

$\hat{\Phi} = \Phi$ flux operator

$\hat{Q} = -i\hbar \frac{\partial}{\partial \Phi}$ charge operator

Hamiltonian Operator for Electrical Harmonic Oscillator

$$\hat{H} = \frac{\hat{Q}^2}{2C} + \frac{\hat{\Phi}^2}{2L} \qquad (*)$$

or

$$\hat{H} = \hbar\omega\left(\hat{a}^+\hat{a} + \frac{1}{2}\right) \qquad (**)$$

with creation and annihilation operators

$$\hat{a}^+ = \frac{1}{\sqrt{2\hbar Z_C}}\left(Z_C\hat{Q}^+ - i\,\hat{\Phi}^+\right)$$

$$\hat{a} = \frac{1}{\sqrt{2\hbar Z_C}}\left(Z_C\hat{Q} + i\,\hat{\Phi}\right)$$

with $Z_C = \sqrt{\frac{L}{C}}$ impedance of oscillator

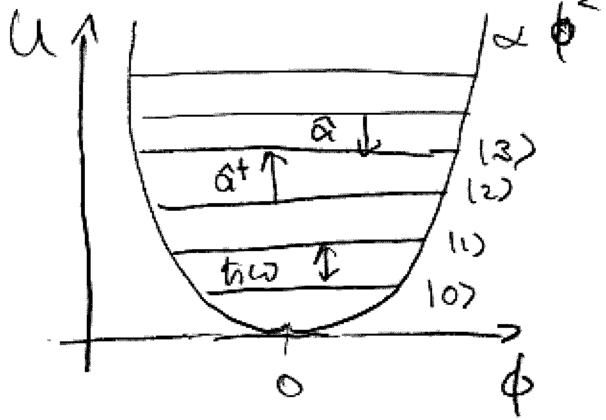
⇒ substitute $\hat{a}^+$ and $\hat{a}$ into (**) to find (*) making use of the commutation relation $[\hat{\Phi}, \hat{Q}] = i\hbar$

- properties of creation and annihilation operators

$$\hat{a}^+|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

$$\hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle$$

$$\hat{a}^+\hat{a}\,|n\rangle = n\,|n\rangle$$



- charge and flux operators expressed in terms of creation and annihilation operators

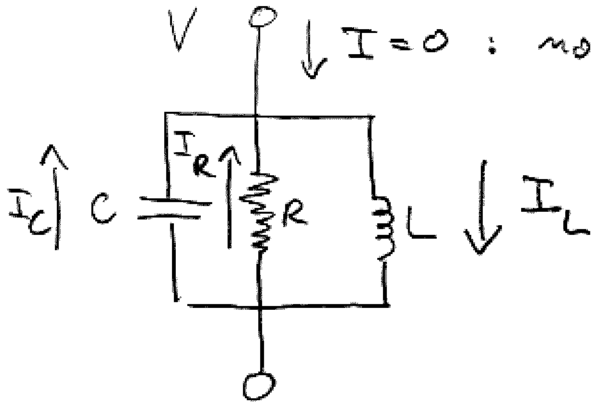
$$\hat{Q} = \sqrt{\frac{\hbar}{2Z_C}}\left(\hat{a} + \hat{a}^+\right)$$

⇝ relates to electric field stored in capacitor

$$\hat{\Phi} = \sqrt{\frac{2Z_C}{\hbar}}\left(\hat{a} - \hat{a}^+\right)$$

⇝ relates to magnetic field stored in inductor

Damped harmonic oscillator



$\downarrow I = 0$: no source connected

$$I_L = I_R + I_C$$

Kirchoff - law

- Curret through resistor $I_R = \frac{V}{R}$
- displacement current $I_C = \dot{Q}_C = C\dot{V}$
- voltage across inductor $V = -L\dot{I}_L$

$$-I_C - I_R + I_L = 0$$

$$\Leftrightarrow \; -C\dot{V} - \frac{V}{R} + I_L = 0$$

$$\Leftrightarrow \; LC\ddot{I}_L + \frac{L}{R}\dot{I}_L + I_L = 0$$

$$\Leftrightarrow \; \boxed{\ddot{I}_L + \frac{1}{RC}\dot{I}_L + \frac{1}{LC}I_L = 0}$$

differential equation for current

solution $I_L(t) = I_L(0)\, e^{\lambda t}$

$$\lambda_{1,2} = \frac{1}{2LC}\left(-\frac{L}{R} \pm \sqrt{\underbrace{\left(\frac{L}{R}\right)^2 - 4LC}_{<0 \text{ and } 4LC \gg \frac{L}{R}}}\right)$$

$\Rightarrow$ underdamped oscillator

$$= \underbrace{-\frac{1}{2RC}}_{\text{decay constant } \alpha} \pm i \underbrace{\frac{1}{\sqrt{LC}}}$$

oscillation frequency $\omega_r = \frac{1}{\sqrt{LC}}$

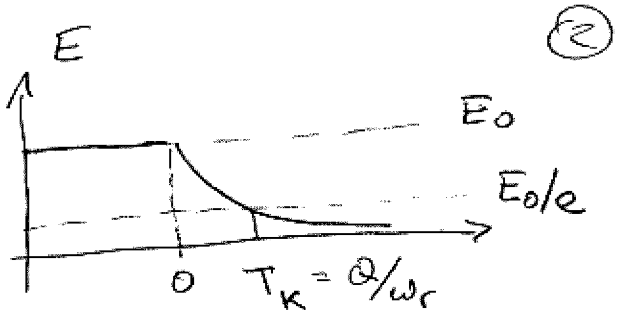
- decay constant $\alpha = -\frac{1}{2RC} = \frac{1}{\tau} \Rightarrow \tau = 2RC$
- energy decay $E \propto \frac{1}{2}LI_L^2 \propto e^{-\frac{1}{RC}t}$

$$\boxed{T_K = RC}$$

energy decay time

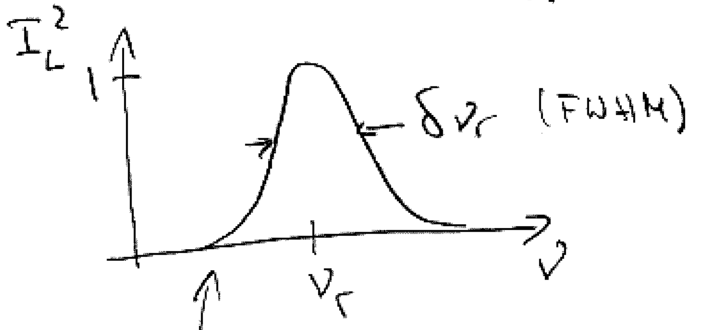
- energy decay rate:

$$K = \frac{1}{T_K} = \frac{1}{RC}$$



- quality factor:

$$Q = \omega_r T_K = \frac{\omega_r}{K} = \frac{\nu_r}{\delta\nu_r}$$



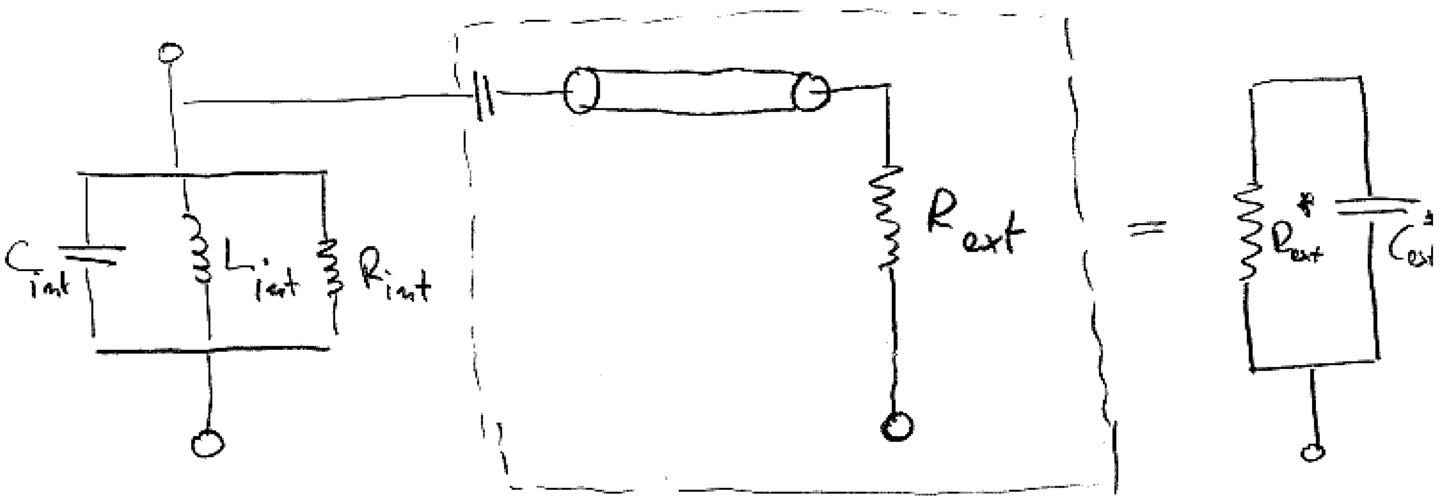
(current amplitude)^2 vs. oscillation frequency

Lorentzian line shape

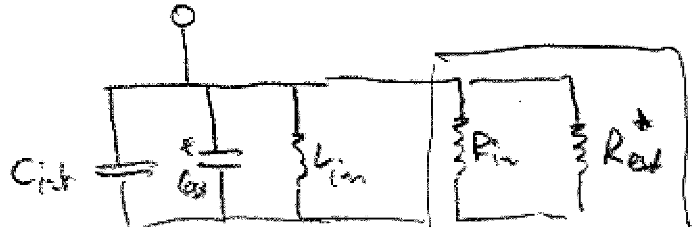
$$I_L^2 \propto I_L(0)^2 \frac{\delta\nu_r/\pi}{(\nu-\nu_r)^2+\delta\nu_r^2}$$

describes spectral response of harmonic oscillator with dissipation

- internal and external dissipation



⇒



$$\boxed{\frac{1}{R} = \frac{1}{R_{int}} + \frac{1}{R_{ext}^*}}$$

$T_K = RC$

The Josephson junction as a non-linear inductor

induction law $V = -L\dot{I}$

Josephson equations: $I = I_0 \sin\delta$ (dc-equation)

$$\dot{I} = I_0 \cos\delta\, \dot{\delta}$$

$$V = \frac{\Phi_0}{2\pi}\dot{\delta}$$

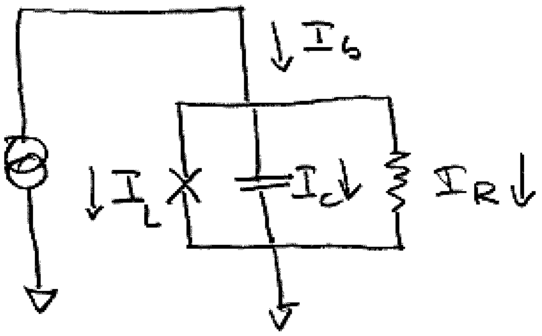
$$= \underbrace{\frac{\Phi_0}{2\pi}\frac{1}{I_0\cos\delta}}_{L_J:\ \text{Josephson inductance}}\dot{I}$$

$$L_J = \underbrace{\frac{\Phi_0}{2\pi I_0}}_{L_J^0\ \Rightarrow\ \text{specific Josephson inductance}}\frac{1}{\cos\delta} \quad \leftarrow \text{non-linearity}$$

$$\boxed{\delta = 2\pi\frac{\Phi}{\Phi_0}}$$

The current biased Josephson junction as a qubit

27.10.2008



Josephson relations:

$$I_L = I_0 \sin\delta$$
$$V = \frac{\Phi_0}{2\pi}\dot{\delta}$$

$$I_C = \dot{Q} = C\dot{V} = C\frac{\Phi_0}{2\pi}\ddot{\delta}$$
$$I_R = \frac{V}{R} = \frac{\Phi_0}{2\pi}\frac{1}{R}\dot{\delta}$$

$$I_b = I_C + I_R + I_L$$

$$\Rightarrow C\frac{\Phi_0}{2\pi}\ddot{\delta} + \frac{\Phi_0}{2\pi}\frac{1}{R}\dot{\delta} + I_0\sin\delta - I_b = 0$$

$\rightarrow \times \frac{\Phi_0}{2\pi}$

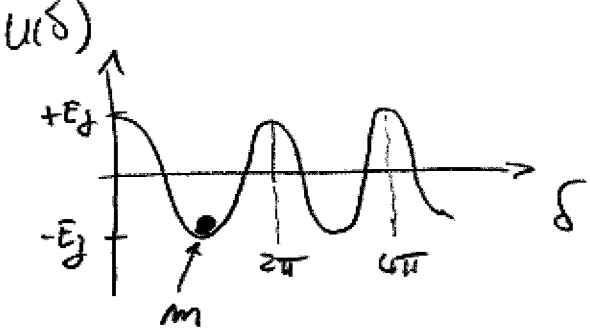
$$\Rightarrow \underbrace{m\ddot{\delta}}_{\text{acceleration}} + \underbrace{m\frac{1}{RC}\dot{\delta}}_{\text{damping with time constant } RC} + \underbrace{\frac{I_0\Phi_0}{2\pi}\left(\sin\delta - \frac{I_b}{I_0}\right)}_{\frac{\partial U(\delta)}{\partial\delta}} = 0$$

$\frac{\partial U(\delta)}{\partial\delta}$: force on phase particle

$\Rightarrow$ equation of motion for current biased Josephson junction

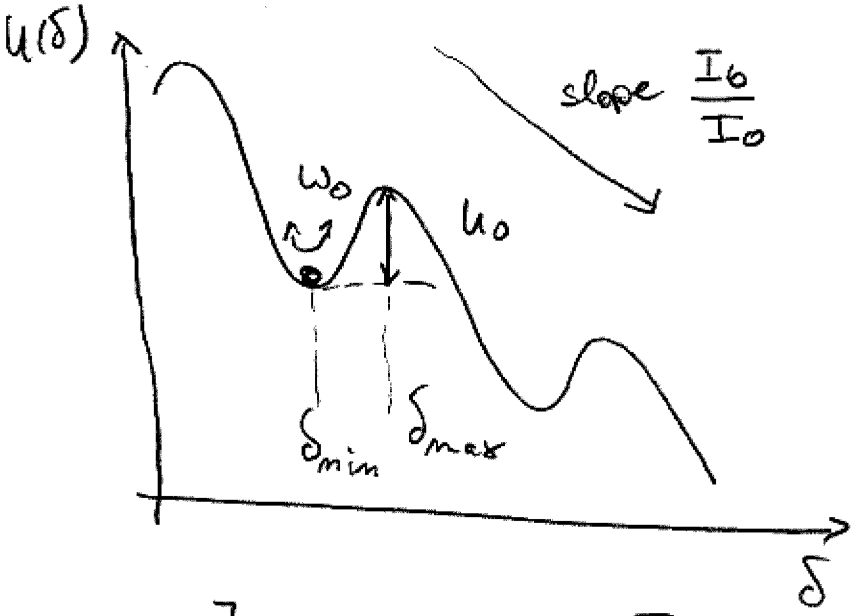
- particle mass $m = C\left(\frac{\Phi_0}{2\pi}\right)^2$
- potential $U(\delta) = -\frac{I_0\Phi_0}{2\pi}\left(\cos\delta + \frac{I_b}{I_0}\delta\right)$
- damping constant $\alpha = \frac{1}{RC}$
- Josephson energy $E_J = \frac{I_0\Phi_0}{2\pi}$

Dynamics of phase particle in eff. potential of Josephson junction



$I_b = 0$: $U(\delta) = E_J \cos\delta$

$I_b < I_0$:



- height of potential barrier

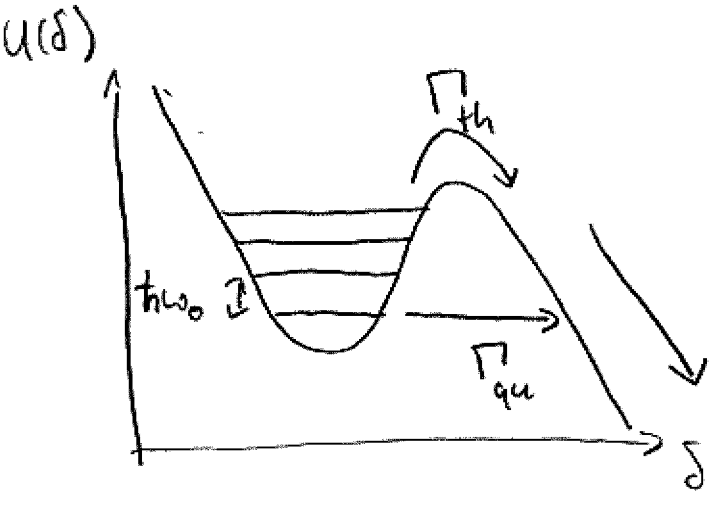
$$U_0 = 2E_J\left[\sqrt{1-\gamma^2} - \gamma\arccos\gamma\right] \quad \text{with} \quad \gamma = \frac{I_b}{I_0}$$

⇒ calculate from $U(\delta)$ finding $U(\delta_{max}) - U(\delta_{min})$

- oscillation frequency of particle
  ↳ bottom of well

$$\omega_0 = \sqrt{\frac{U''(\delta_{min})}{m}} = \omega_p(1-\gamma^2)^{1/4} \quad \text{with} \quad \omega_p = \sqrt{\frac{I_c\, 2\pi}{c\, \Phi_0}}$$

plasma frequ



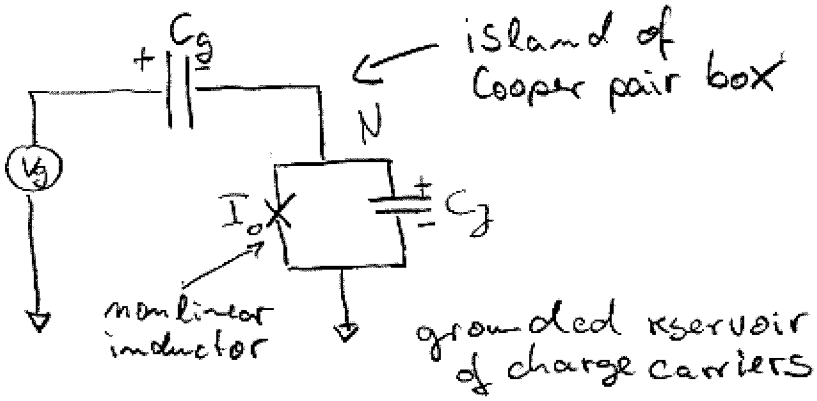
$\underbrace{\dot{\delta} \propto V > 0}$

↳ detection of tunneling

Γ_{th}, Γ_{qu} depend on U_0 and ω_0

27.10.2008

Cooper Pair Box Qubits



charge carriers are Cooper pairs

$$N = \frac{Q}{2e}$$

Hamiltonian

$$H = \underbrace{\frac{Q^2}{2C}}_{\text{electro-static energy}} + \underbrace{\frac{\Phi^2}{2L}}_{\text{magnetic energy}}$$

- charge on island:

$$Q = 2e\,(N - N_g)$$

with polarisation charge

$$N_g = \frac{C_g V_g}{2e}$$

- electrostatic energy

$$H_{el} = \frac{Q^2}{2C} = \underbrace{\frac{(2e)^2}{2C_\Sigma}}_{= E_c \text{ charging energy}} (N - N_g)^2$$

- total capacitance of island

$$C_\Sigma = C_J + C_g$$

- magnetic energy

$$H_{mag} = -E_J \cos\delta = -\overbrace{\frac{\Phi_0 I_0}{2\pi}}^{\text{Josephson Energy}} \cos\delta \qquad \delta \qquad \text{with } \cos\delta \approx 1 - \frac{\delta^2}{2} + \ldots$$

$$\approx -\frac{\Phi_0 I_0}{2\pi}\left(1 - \frac{1}{2}\left(\overbrace{\frac{\Phi}{\Phi_0}2\pi}^{}\right)^2 + \ldots\right)$$

$$= \underbrace{\frac{1}{2}\frac{\Phi^2}{L_J}}_{\text{standard expression for magnetic energy}} \qquad L_J \sim \frac{\Phi_0}{2\pi I_0}$$

- Hamiltonian

$$\hat{H} = E_c(\hat{N} - N_g)^2 - E_J \cos\hat{\delta} \qquad [\hat{\delta}, \hat{N}] = i$$

$$\hat{H} = E_C(\hat{N} - N_g)^2 - E_J \frac{1}{2}\left(e^{i\hat{\delta}} + e^{-i\hat{\delta}}\right)$$

<u>charge basis:</u>

$$[\hat{\delta}, \hat{N}] = i \quad \Rightarrow \quad e^{\pm i\hat{\delta}} |N\rangle = |N \pm 1\rangle$$

$$\hat{N} |N\rangle = N |N\rangle$$

$$\sum_N |N\rangle\langle N| = I$$

$$\langle M|N\rangle = \delta_{MN}$$

Hamiltonian in charge basis

$$\hat{H} = \sum_N \left(\underbrace{E_C (N - N_g)^2 |N\rangle\langle N|}_{\text{energy of charges on island}} - \frac{E_J}{2} \underbrace{\left(|N\rangle\langle N+1| + |N+1\rangle\langle N|\right)}_{\text{energy to add or remove charges from island}} \right)$$

Solve Hamiltonian by diagonalizing in the discrete charge basis

$$\hat{H}|\psi\rangle = E|\psi\rangle$$

or in the continuous and periodic phase basis

$$\hat{N} = \frac{\hat{Q}}{2e} = -i\hbar \frac{1}{2e} \frac{\partial}{\partial \phi} = -i \frac{\partial}{\partial \delta}$$

$$\hat{H} = E_C \left(-i \frac{\partial}{\partial \delta} - N_g\right)^2 - E_J \cos\delta$$

⇒ - Hamiltonian is exactly solvable in δ

- solutions are characteristic Mathieu-functions

3.11.2008

Quantum voltage fluctuations of harmonic oscillator

- voltage operator : $$V = \sqrt{\frac{\quad}{2C}}\,(a\ + a)$$ position

  flux operator $$\hat{\Phi} = \sqrt{\frac{\hbar\omega_r}{2L}}\,(\hat{a}^{+} - \hat{a})$$ "momentum"

- voltage across oscillator for $n = 0$ (expectation value)

$$\langle 0|\hat{V}|0\rangle = 0$$

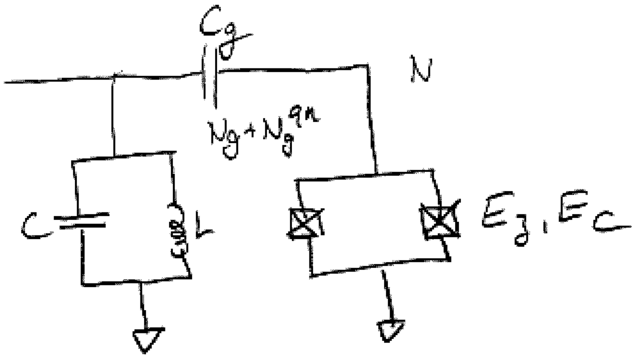
fluctuations of voltage

$$\Delta V_0 = \sqrt{\langle 0|\hat{V}^2|0\rangle}$$

$$= \sqrt{\frac{\hbar\omega_r}{2C}}\,\sqrt{\underbrace{\langle 0|\,\hat{a}^{+}\hat{a}^{+} + \hat{a}^{+}\hat{a} + \hat{a}\hat{a}^{+} + \hat{a}\hat{a}\,|0\rangle}_{=1}}$$

The Jaynes-Cummings Hamiltonian for Circuit QED

- calculate the coupling between Cooper pair box qubit and vacuum fluctuations in harmonic oscillator



- qubit Hamiltonian

$$\hat{H} = \frac{E_C}{2}\big(\overbrace{1-2(N_g}^{0}+\hat{N}_g^{qu})\big)\hat{\sigma}_z - \frac{E_J}{2}\hat{\sigma}_x$$

simplify $N_g = \frac{1}{2}$; $\hat{\sigma}_z \rightarrow \hat{\sigma}_x$; $\hat{\sigma}_x \rightarrow -\hat{\sigma}_z$ change of basis

$$= \frac{E_C}{2}\frac{C_g}{2e}\underbrace{\sqrt{\frac{\hbar\omega_r}{2C}}(\hat{a}^+ + \hat{a})}_{\hat{V}}\underbrace{\overset{}{\hat{\sigma}_x}}_{\|\,(\hat{\sigma}^+ + \hat{\sigma}^-)} + \frac{E_J}{2}\hat{\sigma}_z$$

$$\cancel{\hat{a}^+\hat{\sigma}^+} + \hat{a}^+\hat{\sigma}^- + \hat{a}\hat{\sigma}^+ + \cancel{\hat{a}\hat{\sigma}^-} \quad \underline{RWA}$$

- full Circuit QED hamiltonian

$$\hat{H} = \hbar\omega_r\left(\hat{a}^+\hat{a} + \frac{1}{2}\right) + \frac{E_J}{2}\hat{\sigma}_z + \underbrace{\frac{C_g}{C_\Sigma}2e\sqrt{\frac{\hbar\omega_r}{2C}}}_{\hbar g}(\hat{a}^+\hat{\sigma}^- + \hat{a}\hat{\sigma}$$

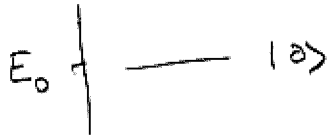
$$\Uparrow$$

$\frac{2g}{\pi}$ = Vacuum - Rabi frequency

Sources of Decoherence:

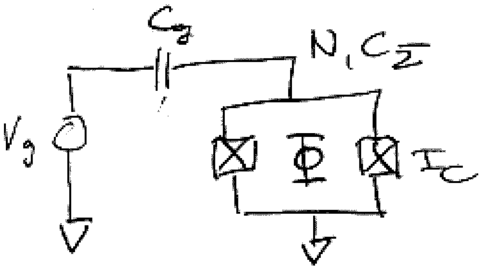
24.11.2008 ①

on parameters λ of the hamiltonian



$$\underline{\lambda}: \quad E_c = \frac{(2e)^2}{C_\Sigma} \; ; \; N_g = \frac{C_g V_g}{2e}$$

$$E_J = \frac{I_c \Phi_0}{2\pi} \; ; \; \delta = \pi \frac{\Phi}{\Phi_0}$$



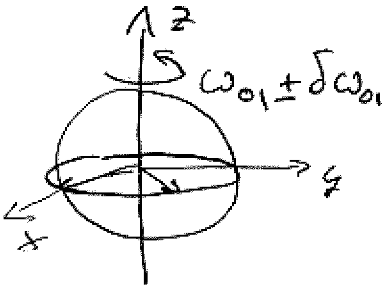
- fluctuations in parameter λ of hamiltonian lead to fluctuation in energy of qubit states and thus to fluctuations of phase of qubit (e.g. in a superposition state)

$$\hat{H} = H_0 \hat{\sigma}_z$$

$$\hat{H} = \left(H_0(\lambda_0) + \frac{\partial H_0}{\partial \lambda} \delta\lambda + \frac{\partial^2 H_0}{\partial \lambda^2} \delta\lambda^2 + \dots \right) \hat{\sigma}_z$$

δλ : deviation from desired parameter λ_0

- $$\Delta\phi = \frac{\partial \omega_{01}}{\partial \lambda} \int_0^t \delta\lambda(t')\, dt'$$



deviation $\Delta\phi$ of phase from desired value ϕ_0 after time t

- How to avoid fluctuations $\Delta\phi$ in phase?

  - $\delta\lambda = 0$ avoid fluctuations in parameter
  - $\frac{\partial H_0}{\partial \lambda} = 0$ avoid sensitivity of Hamiltonian to parameter
  - spin echo: dynamically cancel the effect of slow fluctuations

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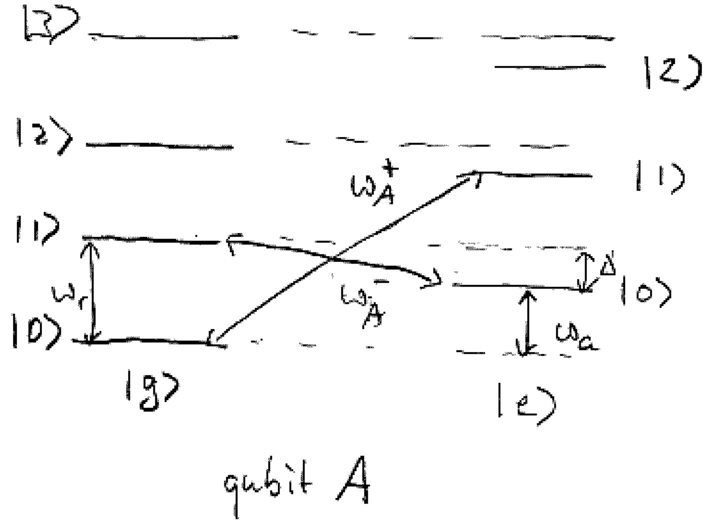
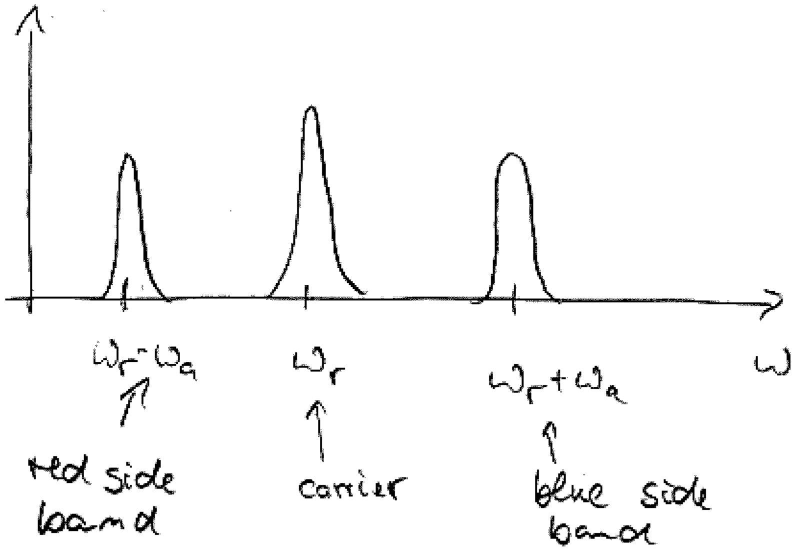
Side Band Transitions in a Superconducting Qubit Coupled to a Resonator

- dressed states energy level diagram for detuned qubit $|\omega_a - \omega_r| = |\Delta| \ll g$



- blue side band $\omega_A^+ = \omega_r + \omega_a$ ← Creates both an excitation in the resonator and in the qubit
- red side band $\omega_A^- = \omega_r - \omega_a$ ↖ takes an excitation out of the resonator and puts it into the qubit



⇝ similar ideas are realized in ion-traps and in other coupled qubit/oscillator systems

Generating Entanglement using Side Band

- consider 3 quantum systems

qubit A, qubit B, resonator

$$|gg0\rangle \xrightarrow{\pi_A} |eg0\rangle$$

$$\xrightarrow{(\pi/2)^+_B} \frac{1}{\sqrt{2}}\left(|eg0\rangle + |ee1\rangle\right)$$

$$\xrightarrow{(\pi)^+_A} \frac{1}{\sqrt{2}}\left(|eg0\rangle + |ge0\rangle\right)$$

$$= \underbrace{\frac{1}{\sqrt{2}}\left(|eg\rangle + |ge\rangle\right)}_{\psi^+ \text{ - Bell state}} \otimes |0\rangle$$

$$\xrightarrow{\pi_B} \underbrace{\frac{1}{\sqrt{2}}\left(|ee\rangle + |ge\rangle\right)}_{\phi^+ \text{ - Bell state}} \otimes |0\rangle$$

Cirac - Zoller CNOT gate

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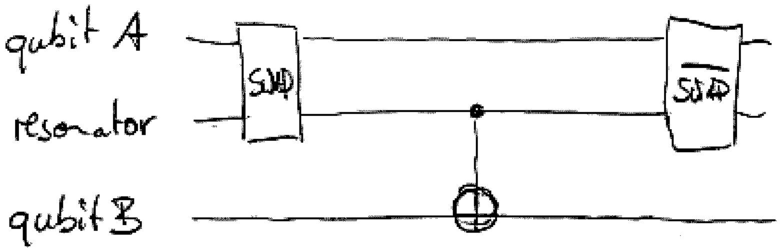
- general idea:



① swap state of qubit A into resonator

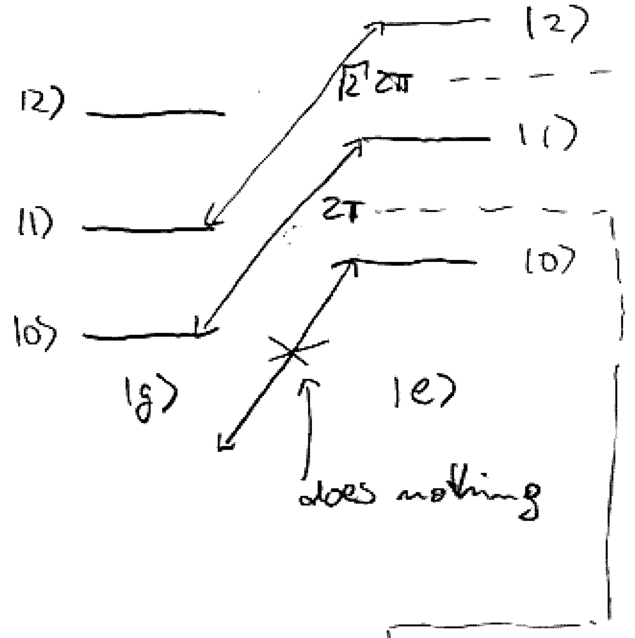
$$\frac{1}{\sqrt{2}}\left(|g\rangle+|e\rangle\right)\otimes|0\rangle \xrightarrow{\pi^{+}} \frac{1}{\sqrt{2}}\left(|e,1\rangle+|e,0\rangle\right)$$

$$= |e\rangle\otimes\underbrace{\frac{1}{\sqrt{2}}\left(|0\rangle+|1\rangle\right)}_{\text{resonator has amplitudes and phases of qubit}}$$

② perform CNOT operation between qubit B and resonator

③ swap resonator state back into qubit A

- phase gate



interaction is by a factor of $\sqrt{n}$ stronger for side bands starting from an n photon state

- starting in $|g0\rangle$: blue side band induces Rabi oscillations between $|g0\rangle$ and $|e1\rangle$
- starting in $|g1\rangle$:

  $|g1\rangle \rightarrow |e2\rangle$

  leaving qubit A Hilbert space!

Problem!

2π rotation induces a global phase factor in any two-level system

- Unitary of phase gate

$$\hat{U}_\phi = \begin{array}{cc} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} & \begin{array}{c} |e0\rangle \\ |g0\rangle \\ |e1\rangle \\ |g1\rangle \end{array} \\ \begin{array}{cccc} |e0\rangle & |g0\rangle & |e1\rangle & |g1\rangle \end{array} & \end{array}$$

- Trick to avoid populating states with $n > 1$
  - composite rotations

$|g,0\rangle$: $\left(\pi/\sqrt{2}\right)^+_y \longrightarrow \left(\pi\right)^+_x \longrightarrow \left(\pi/\sqrt{2}\right)^+_y \longrightarrow \left(\pi\right)^+_x \rightarrow -|g0\rangle$

$|g,1\rangle$: $\left(\pi\right)^+_y \longrightarrow \left(\pi\sqrt{2}\right)^+_x \rightarrow \left(\pi\right)^+_y \longrightarrow \left(\pi\sqrt{2}\right)^+_x \rightarrow -|g1\rangle$

Complete Circuit for CNOT gate sequence