Dispersive limit: $H = H_0 + \lambda H_1$ λ small

17.11.2008

Formula: $e^{\lambda S} H e^{-\lambda S} = H + \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} [S, H]_{(n)}$ with

$$[S,H]_{(n)} = [S, [S,H]_{(n-1)}]$$
$$[S,H]_{(1)} = [S,H]$$

$$= H_0 + \lambda H_1 + \lambda [S,H] + \frac{\lambda^2}{2} [S,[S,H]] + \mathcal{O}(\lambda^3)$$

choose S such that $[S, H_0] = -H_1$ (Schrieffer-Wolff-Transf.)

$$\Rightarrow \underline{H_0 + \lambda H_1 + (-\lambda H_1) + \lambda^2 [S, H_1]} + \frac{\lambda^2}{2} [S, -H_1 + \lambda [S, H_1]] + \mathcal{O}(\lambda^3)$$

$$\boxed{= H_0 - \frac{\lambda^2}{2} [S, H_1] + \mathcal{O}(\lambda^3)}$$

e.g.: CPB Hamiltonian: $H_0 = \frac{\hbar\Omega}{2} \sigma_z$ with $\Omega = \sqrt{E_c^2 + E_J^2}$

const. Voltage on Resonator: $\varepsilon \sigma_x$

$$H = H_0 + \varepsilon \sigma_x$$

$[S, H_0] = -H_1$: $[S, \frac{\hbar\Omega}{2}\sigma_z] = -\sigma_x$: $S = \frac{i}{\hbar\Omega}\sigma_y$

$$\Rightarrow \tilde{H} = H_0 - \frac{\varepsilon^2}{2}\left[\frac{i}{\hbar\Omega}\sigma_y, \sigma_x\right] = H_0 - \frac{\varepsilon^2}{\hbar\Omega}\sigma_z \quad \ldots \text{ small energy shift}$$

e.g. Jaynes-Cummings Hamiltonian in dispersive limit

$$H = H_0 + \hbar g \underbrace{(a\sigma^+ + a^\dagger\sigma^-)}_{H_1} \qquad H_0 = \hbar\omega_1 (a^\dagger a + \tfrac{1}{2}) + \frac{\hbar\Omega}{2}\sigma_z$$

$[S, H_0] = -H_1$: works with: $S = \beta (a\sigma^+ - a^\dagger \sigma^-)$

$$[S, H_0] = \hbar\omega_1 \beta [a\sigma^+ - a^\dagger\sigma^-, a^\dagger a] + \frac{\hbar\Omega}{2}\beta [a\sigma^+ - a^\dagger\sigma^-, \sigma_z] =$$
$$= \hbar\omega_1 \beta (\sigma^+ [a, a^\dagger a] - \sigma^- [a^\dagger, a^\dagger a]) + \frac{\hbar\Omega}{2}\beta (a [\sigma^+, \sigma_z] - a^\dagger [\sigma^-, \sigma_z])$$

$$[a, a^\dagger a] = a a^\dagger a - a^\dagger a a = \left| [a, a^\dagger] = 1 \right| = (1 + a^\dagger a) a - a^\dagger a a = a$$

$$[a^\dagger, a^\dagger a] = a^\dagger a^\dagger a - a^\dagger a a^\dagger = a^\dagger a^\dagger a - a^\dagger (1 + a^\dagger a) = - a^\dagger$$

$$[\sigma^\pm, \sigma_z] = \frac{1}{2}\left\{[\sigma_x, \sigma_z] \pm i [\sigma_y, \sigma_z]\right\} = \left| [\sigma_i, \sigma_j] = 2 i \epsilon_{ijk} \sigma_k \right| =$$

$$= \frac{1}{2}\left\{2i(-\sigma_y) \pm 2 i^2 \sigma_x\right\} = \frac{1}{2}\left\{-2(\pm\sigma_x + i\sigma_y)\right\} = \mp 2 \sigma^\pm$$

$$\Rightarrow \hbar\omega_1 \cdot \beta\,(\sigma^+ a + \sigma^- a^\dagger) + \frac{\hbar\Omega}{2}\beta\left((-2) a\sigma^+ - 2 a^\dagger \sigma^-\right) =$$

$$= \hbar(\omega_1 - \Omega)\,\beta\,[\sigma^+ a + \sigma^- a^\dagger] = -H_1 \quad \text{for } \beta = -\frac{1}{\hbar(\omega_1 - \Omega)} =$$

$$= \frac{1}{\hbar\Delta} \qquad \Delta = (\Omega - \omega_1)$$

$$\Rightarrow S = \frac{1}{\hbar\Delta}(a\sigma^+ - a^\dagger\sigma^-)$$

$$e^{\lambda S} = e^{\hbar g S} = e^{\frac{g}{\Delta}(a\sigma^+ - a^\dagger \sigma^-)}$$

$$[S, H_1] = \frac{1}{\hbar\Delta}\left[a\sigma^+ - a^\dagger\sigma^-, a\sigma^+ + a^\dagger\sigma^-\right] =$$

$$= \frac{1}{\hbar\Delta}\underbrace{[a\sigma^+, a\sigma^+]}_{0} \underbrace{- [a^\dagger\sigma^-, a\sigma^+] + [a\sigma^+, a^\dagger\sigma^-]}_{} - \underbrace{[a^\dagger\sigma^-, a^\dagger\sigma^-]}_{0}$$

$$2[a\sigma^+, a^\dagger\sigma^-] =$$

$$= 2(a a^\dagger \sigma^+\sigma^- - a^\dagger a \sigma^-\sigma^+) =$$

$$= 2(\sigma^+\sigma^- + a^\dagger a \sigma^+\sigma^- - a^\dagger a \sigma^-\sigma^+) =$$

$$= 2(\sigma^+\sigma^- + a^\dagger a \underbrace{[\sigma^+, \sigma^-]}_{\sigma_z}) = 2(\sigma^+\sigma^- + a^\dagger a \sigma_z)$$

$$\sigma^+\sigma^- = \frac{1}{4}(\sigma_x + i\sigma_y)(\sigma_x - i\sigma_y) = \frac{1}{4}(\sigma_x^2 + \sigma_y^2 + i[\sigma_y, \sigma_x]) = \frac{1}{4}(2\mathbb{1} - 2i^2\sigma_z) =$$

$$= \frac{1}{2}(\mathbb{1} + \sigma_z)$$

$$\tilde{H} = H_0 - \underbrace{\frac{(\hbar g)^2}{\hbar\Delta}}_{\text{energy zero shift}}\left(\cancel{\tfrac{1}{2}\mathbb{1}} + \tfrac{1}{2}\sigma_z + a^\dagger a \sigma_z\right) = \boxed{\hbar\left(\omega_1 + \frac{g^2}{\Delta}\sigma_z\right) a^\dagger a + \frac{\hbar}{2}\left(\Omega + \frac{g^2}{\Delta}\right)\sigma_z}$$

Rotating Wave approximation:

apply time dependent signal to CPB by changing N_g
(classical coherent signal)

$\hbar = 1$

$$H = -\frac{E_{el}(t)}{2}\sigma_z - \frac{E_J}{2}\sigma_x \qquad E_{el}(t) = E_C\left(1 - 2\left[N_g(0) + \eta \cdot \cos\omega t\right]\right)$$

$$= -\frac{E_{el}(0)}{2}\sigma_z - \frac{E_J}{2}\sigma_x + \overbrace{2E_C\eta}^{\varepsilon}\cos\omega t\,\sigma_z$$

in Eigenbasis at $t=0$ for $N_g(0) = \frac{1}{2}$:

$$H = \frac{\Omega}{2}\sigma_z + \varepsilon\cos\omega t\,\sigma_x = \vec{m}(t)\cdot\vec{\sigma} \qquad \vec{m}(t) = \begin{pmatrix}\varepsilon\cos\omega t\\ 0\\ \Omega/2\end{pmatrix}$$

Larmor Precession of unperturbed system: $|\psi(t)\rangle = e^{-i\frac{\Omega}{2}\sigma_z t}\,|\psi(0)\rangle$

Take the component of $\vec{m}(t)$ which rotates at approx. the Larmor frequency:

$$\vec{m}(t) = \frac{1}{2}\begin{pmatrix}\varepsilon\cos\omega t\\ \varepsilon\sin\omega t\\ \Omega/2\end{pmatrix} + \frac{1}{2}\begin{pmatrix}\varepsilon\cos\omega t\\ -\varepsilon\sin\omega t\\ \Omega/2\end{pmatrix}$$

$$H = \frac{\Omega}{2}\sigma_z + \frac{\varepsilon}{2}\overbrace{\left[\cos\omega t\,\sigma_x + \sin\omega t\,\sigma_y\right]}^{①^+} + \frac{\varepsilon}{2}\overbrace{\left[\cos\omega t\,\sigma_x - \sin\omega t\,\sigma_y\right]}^{①^-}$$

Rotating Frame $U = e^{i\frac{\tilde{\omega}t}{2}\sigma_z}$ $\qquad UHU^\dagger \underbrace{U|\psi\rangle}_{|\phi\rangle} = i\hbar\, U\frac{d}{dt}(U^\dagger \underbrace{U|\psi\rangle}_{|\phi\rangle}) = i\hbar(U\dot{U}^\dagger + \frac{d}{dt}|\phi\rangle)$

$$\tilde{H} = UHU^\dagger - i\,U\dot{U}^\dagger: \qquad U\sigma_z U^\dagger = \sigma_z$$

$$U\sigma_x U^\dagger = \begin{pmatrix}e^{\frac{i\tilde{\omega}t}{2}} & 0\\ 0 & e^{-\frac{i\tilde{\omega}t}{2}}\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}e^{\frac{i\tilde{\omega}t}{2}} & 0\\ 0 & e^{-\frac{i\tilde{\omega}t}{2}}\end{pmatrix}.$$

$$= \cos\tilde{\omega}t\,\sigma_x - \sin\tilde{\omega}t\,\sigma_y$$

$$U\sigma_y U^\dagger = \sin\tilde{\omega}t\,\sigma_x + \cos\tilde{\omega}t\,\sigma_y$$

$$U\dot{U}^\dagger = -i\,\tilde{\omega}/2\,\sigma_z$$

$①^\pm$: $\cos\omega t\left(\cos\tilde{\omega}t\,\sigma_x - \sin\tilde{\omega}t\,\sigma_y\right) \pm \sin\omega t\left(\sin\tilde{\omega}t\,\sigma_x + \cos\tilde{\omega}t\,\sigma_y\right) =$

$$\tilde{\omega} = \omega = \begin{cases} +: & (\cos^2\omega t + \sin^2\omega t)\sigma_x = \sigma_x \\ -: & (\cos 2\omega t\,\sigma_x - \sin 2\omega t\,\sigma_y) \text{ - rotating at twice the frequency}\end{cases}$$

$\tilde{H} = \frac{1}{2}(\Omega - \omega)\sigma_z + \frac{\varepsilon}{2}\sigma_x$ => transformation to static Hamiltonian by discarding ①$^-$, which is called the Rotating Wave Approx.

@ resonance for $\omega = \Omega$:

$\tilde{H}_{res} = \frac{\varepsilon}{2}\sigma_x \qquad \tilde{U} = e^{-\frac{i}{\hbar}\tilde{H}_{res}t}$ .. rotation about x-axis

Rotating Wave approximation in Jaynes - Cummings Model:

Rotating Frame Transformation: (Interaction Picture)

$$U = \underbrace{e^{i\frac{\Omega}{2}t\sigma_z}}_{\text{qubit}} \otimes \underbrace{e^{i\omega_r a^+ a}}_{\text{resonator}} = U_q \otimes U_r$$

$$U H_0 U^+ = U\left[\omega_r\left(a^+a + \tfrac{1}{2}\right) + \tfrac{\Omega}{2}\sigma_z\right]U^+ =$$

$$= \omega_r\left[U_r a^+ a U_r^+ + \tfrac{1}{2}\right] + \tfrac{\Omega}{2} U_q \sigma_z U_q^+ = H_0$$

$$-iU\dot{U}^+ = -\frac{\Omega}{2}\sigma_z - \omega_r a^+ a$$

$$\rightarrow \tilde{H}_0 = 0$$

$$hg\, U(a^+ + a)(\sigma^+ + \sigma^-)U^+ = (\underbrace{U_r a^+ U_r^+}_{a^+ e^{i\omega_r t}} + \underbrace{U_r a U_r^+}_{a e^{-i\omega_r t}})(\underbrace{U_q \sigma^+ U_q^+}_{\sigma^+ e^{i\Omega t}} + \underbrace{U_q \sigma^- U_q^+}_{\sigma^- e^{-i\Omega t}})$$

$$\rightarrow a^+\sigma^+ e^{i(\omega_r + \Omega)t} + a^-\sigma^- e^{-i(\omega_r+\Omega)t} +$$

$$+ a\sigma^+ + a^+\sigma^-$$

only those are kept

Bloch equations:

$$H = -\mu\,\vec{\sigma}\cdot\vec{m}(t) \qquad \vec{B}(t) = \begin{pmatrix} m_x \\ m_y \\ m_z \end{pmatrix}$$

$$\dot{\rho} = -\frac{i}{\hbar}[H,\rho] \qquad \rho = |\psi\rangle\langle\psi| = \frac{1}{2}(1 + \vec{r}\cdot\vec{\sigma})$$

$$\dot{\rho} = +\frac{i\mu}{\hbar}\left[\vec{\sigma}\cdot\vec{m}\cdot\frac{1}{2}(1+\vec{r}\vec{\sigma}) - \frac{1}{2}(1+\vec{r}\cdot\vec{\sigma})\cdot\vec{\sigma}\vec{m}\right] =$$

$$= +\frac{i\mu}{2\hbar}\left[(\vec{m}\cdot\vec{\sigma})(\vec{r}\cdot\vec{\sigma}) - (\vec{r}\cdot\vec{\sigma})(\vec{m}\cdot\vec{\sigma})\right]$$

using $(\vec{a}\cdot\vec{\sigma})(\vec{b}\cdot\vec{\sigma}) = (\vec{a}\cdot\vec{b})\,1 + i(\vec{a}\times\vec{b})\cdot\vec{\sigma}$

$$= +\frac{i\mu}{\hbar}\left(-i\,\vec{r}\times\vec{m}\right)\vec{\sigma}$$

$$= \frac{\mu}{\hbar}(\vec{r}\times\vec{m})\cdot\vec{\sigma}$$

in terms of Bloch vector components:

$$\dot{\vec{r}} = Tr(\dot{\rho}\vec{\sigma}) = \overset{\gamma}{\frac{\mu}{\hbar}}(\vec{r}\times\vec{m})$$ ... "Bloch equations"

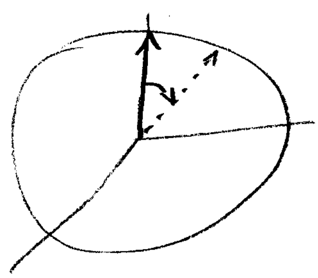
$m_y = 0$

$$\dot{r}_x = \gamma\, r_y m_z$$

$$\dot{r}_y = \gamma(r_z m_x - r_x m_z)$$

$$\dot{r}_z = -\gamma\, r_y m_x$$

e.g.: $\vec{r}(0) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow \dot{r}_y \neq 0$



Problem: no relaxation to thermal equilibrium

e.g. $\vec{m} = m_z$, system in excited state: $(|r_z| = 1) \Rightarrow \dot{r} = 0$

introduce longitudinal and transversal relaxation

$$\dot{r}_x = \gamma (\vec{r} \times m)_x - \frac{r_x}{T_2}$$

$$\dot{r}_y = \gamma (\vec{r} \times m)_y - \frac{r_y}{T_2}$$

$$\dot{r}_z = \gamma (\vec{r} \times m)_z - \frac{r_z - r_z^s}{T_1}$$

r_z^s ... steady state (−1 for ground state)

now: with $r_z^s = -1$: $\dot{r}_z = -(r_z + 1) \cdot \frac{1}{T_1}$

$$\rightarrow r_z(t) = e^{-\frac{t}{T_1}} (r_z + r_z(0)) + 1$$

→ T_1: longitudinal relaxation time

e.g.: $r_x(0) = 1$ $\quad \vec{m} = m_z$; $T_1 = 0$

$$\left.\begin{aligned} \dot{r}_y &= -\gamma r_x m_z - \frac{r_y}{T_2} \\ \dot{r}_x &= \gamma r_y m_z - \frac{r_x}{T_2} \end{aligned}\right\} r_x(t), r_y(t) \propto e^{-\frac{t}{T_2}}$$

T_2: transverse relaxation time, dephasing

rotating frame rate $\omega = \gamma B_z$: $\quad r_x(t) = \cos\omega t\, \tilde{r}_x(t) + \sin\omega t\, \tilde{r}_y(t)$

$\quad r_y(t) = -\sin\omega t\, \tilde{r}_x(t) + \cos\omega t\, \tilde{r}_y(t)$