# Quantum computation in topological Hilbertspaces

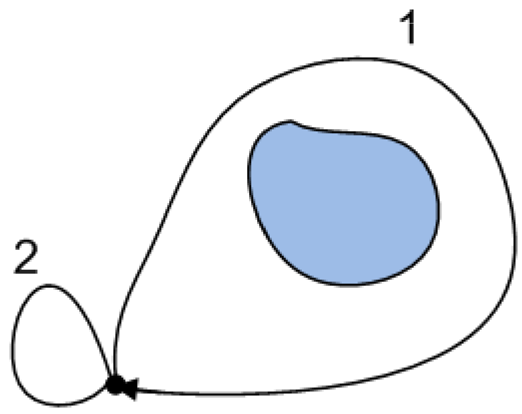
A presentation on topological quantum computing

by Deniz Bozyigit and Martin Claassen

# Introduction

# In two words – what is it about?

- Pushing around fractionally charged quasiparticles in 2D-gases can result in universal gate operations on a Hilbertspace which is accessible by fusioning the particles.



# Historical overview

- Discovery of fractional quantum hall effect (FQHE) by Daniel Tsui, Horst Störmer in 1982
- Observation of fractionally charged particles by R. de-Picciotto in 1997
- Idea of topological quantum computation with anyons **by A. Kitaev in 1997**
- Experimental proof of non-abelian anyons by F. E. Camino, Wei Zhou, V. J. Goldman in 2005
- Names: Kitaev, Preskill, Freedman, Das Sarma

# Presentation plan

- Finding topological degrees of freedom in quantum systems
  - Adiabatic time evolution and Berrys phase
  - The Aharonov-Bohm effect
  - The Cyon anyon
- Generalised anyons
- Comments on errors in topological quantum computers
- Summary

# Adiabatic time evolution and Berrys phase

# Adiabatic time evolution

- Time dependent Hamiltonian

from parameter space $\mathcal{M}$

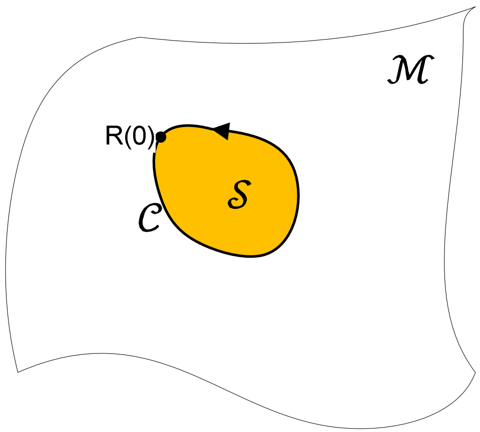
$$H(t) := H(R(t)) = \sum_n E_n(R(t)) |n, R(t)\rangle\langle n, R(t)|$$

Time dependent nth eigenstate

- Considering nontrivial loops in $\mathcal{M}$

$$E_n(R(0)) = E_n(R(T))$$

$$|n, R(0)\rangle \rightarrow |n, R(T)\rangle = e^{i\zeta_n}|n, R(0)\rangle$$




- Time developement of a state:

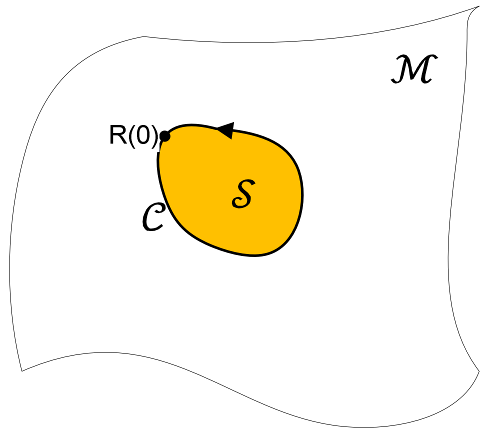
$$\psi(t) = e^{i\alpha_\psi(t)}\psi(0)$$

# Berrys phase and the gauge field

- The phase factor can be decomposed

$$\alpha_\psi(t) = \int_0^t d\tilde{t} E_n(\tilde{t}) + \gamma_n(t)$$

Berrys phase




- For a loop $\mathcal{C}$ in $\mathcal{M}$

$$\gamma_n = \gamma_n(\mathcal{C}) = \oint_{\mathcal{C}} A^n = \int_S dA^n = \int_S F^n$$

Mead-Berry vector potential / gauge potential

$$F^n = i \sum_{m \neq n} \frac{\langle \ldots | \ldots \rangle}{[E_n(R) - E_m(R)]^2}$$

Get's interesting for degenerate subspaces!

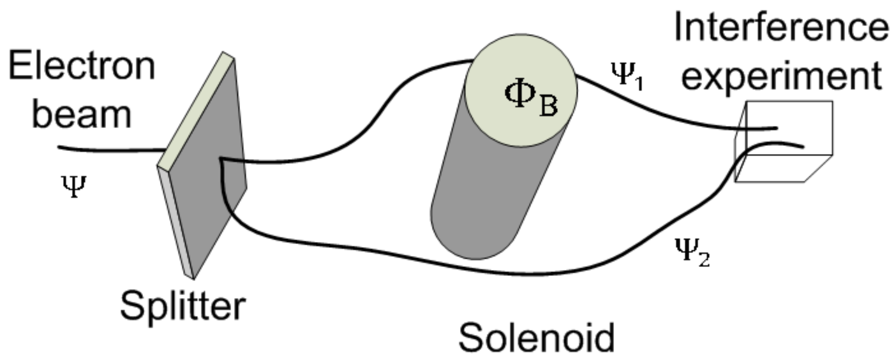
- Berrys phase can be further decomposed

$$\gamma_n = \gamma_{n,path} + \gamma_{n,topo}$$

# Looking for Berrys phase - The Aharonov-Bohm effect

# Experiment and result

1. Splitting a coherent electron beam




2. Passing by an solenoid with flux $\Phi_B$ confined inside

3. Interference experiments with both parts of the beam

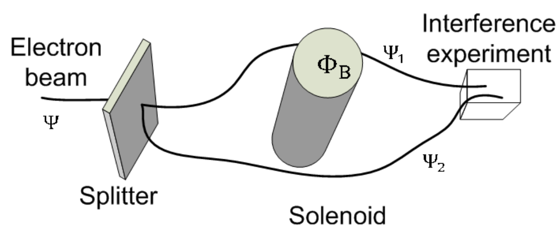
- **Result**: Interference patterns depend on $\Phi_B$

# Looking at Berrys phase

- Beam splits in two parts with different phases

$$\Psi = \Psi_1 + \Psi_2 = e^{-iS_1/\hbar}\Psi_1^0 + e^{-iS_2/\hbar}\Psi_2^0$$

$$\Delta S = S_2 - S_1 = -\frac{e}{c}\oint A^{(el)} \cdot dx = -\frac{e}{c}\Phi_B$$




- Identifying Berrys phase with $\Delta S$

$$\gamma_n = {}^{\Delta S}/_{\hbar} = -\frac{e}{\hbar c}\oint A^{(el)} \cdot dx = -\frac{e}{\hbar c}\Phi_B$$

$$= \gamma_{n,path} + \gamma_{n,topo}$$

Is purely topological in nature for vanishing fields!

Vanishes with the mag. field

# Lessons learned

- Without effective fields, **potentials** can influence quantum mechanical systems!

- The **topological phase** does not change under smooth deformations.

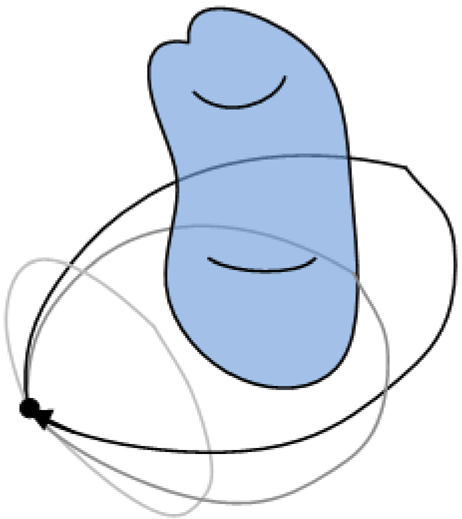
- The **path dependent phase** can disapear for small fields or large distances.

- The two paths are topologically not equivalent, so the topological phase is neither.

# Comment on topology of the A-B effect

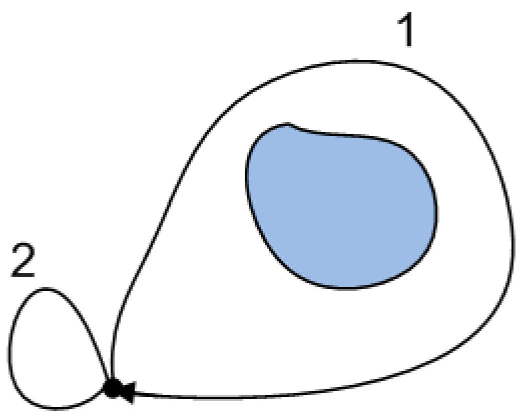
## 3D & higher dimensions

- All loops around one-connected objects are equivalent to trivial loops



## 2D

- Loops around one-connected objects can be non trivial!

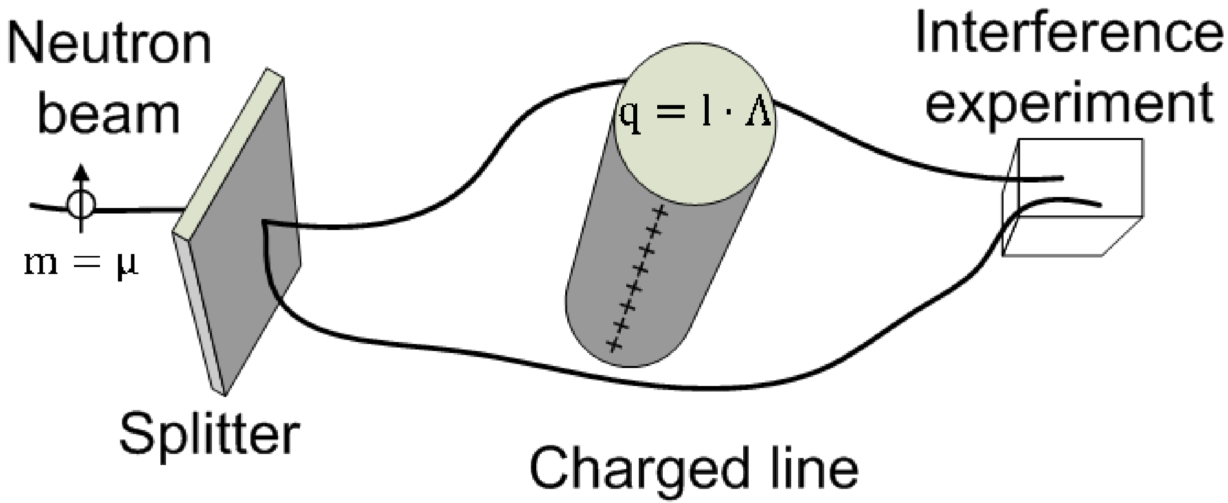


# Aharonov - Casher effect

- A-C is dual to the A-B effect and results also in a Berry type phase shift

$$\gamma_n = \Delta S/\hbar = \pm \frac{4\pi\mu\Lambda}{\hbar c}$$

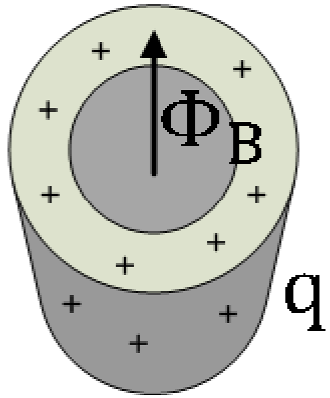
Line charge density



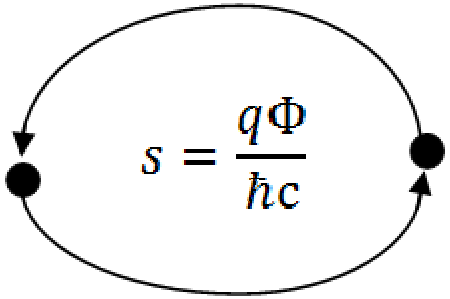
# A mind game –
# The Cyon anyon

# Model of the cyon anyon

- Consider a flux-charge composite particle with flux $\Phi$ and charge $q$ - living in 2D




- One counterclockwise exchange results in an A-B and A-C phase




„Braiding"

$$e^{i(\frac{\theta}{2}+\frac{\theta}{2})} = e^{i\theta} = e^{2\pi i \frac{q\Phi}{\hbar c}} = e^{2\pi i s}$$

As for fermions and bosons!

# Presentation plan

- Finding topological degrees of freedom in quantum systems
- Generalised anyons
  - Components of a model
  - Fusion channels
  - Braiding statistics
  - Physical Interpretation
  - Representation of qubits
- Comments on errors in topological quantum computers
- Summary

# Generalised anyons

# The Anyon Model

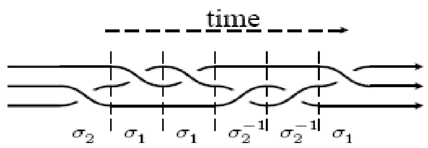
## What do we need for a model description?

1. A list of particle types


2. Rules for fusing and splitting

$$A \times B \rightarrow \sum_{c} N_{ab}^{c} C$$

3. Rules for braiding



# Anyon Model – 1. Particle Types

- A finite set of particle types, distinguished by labels {a,b,c,...}
- Trivial particle: $1 = \bar{1}$
- Antiparticles: $a, \bar{a};\ \ a \times \bar{a} \to 1$

# Anyon Model – 2. Fusion

$$A \times B \rightarrow \sum_c N_{ab}^c C$$

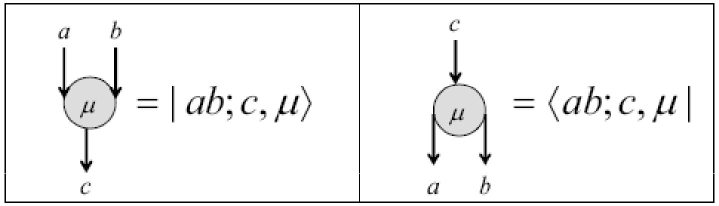
- $N^c_{ab} = 0$ : C cannot be fused from A and B
- $N^c_{ab} = 1$ : A and B can fuse to C
- $N^c_{ab} > 1$ : (A and B can fuse to C in $N^c_{ab}$ distinguishable ways – some models only)

*Properties:*

- **associative**: $A \times B = B \times A$
- **reversible**: $C \longrightarrow A \times B$
- **annihilation**: $N^1_{a\bar{a}} = 1$

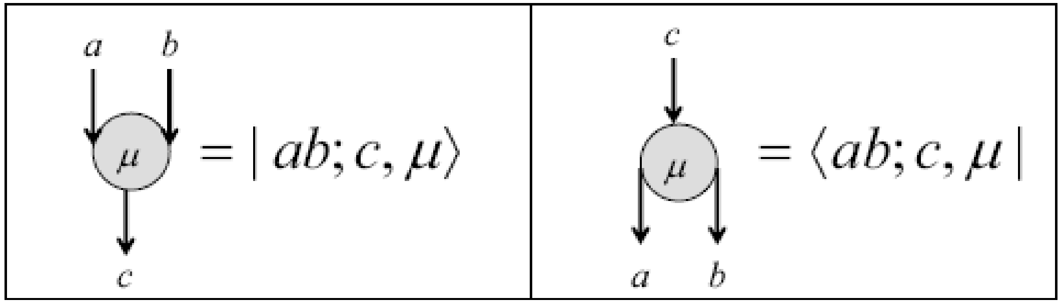
# Anyon Model – 2. Fusion Space (1)

- The $N^c_{ab}$ distinguishable ways of fusing *A* and *B* can be regarded as the orthonormal basis states of a **Hilbert space $V^c_{ab}$**
- We call this space the ***"fusion space"***, and the states it contains the ***"fusion states"***
- Basis elements:




$$\{ |ab;c;\mu\rangle , \qquad \mu = 1,2, \dots , N^c_{ab} \}$$

# Anyon Model – 2. Fusion Space (2)

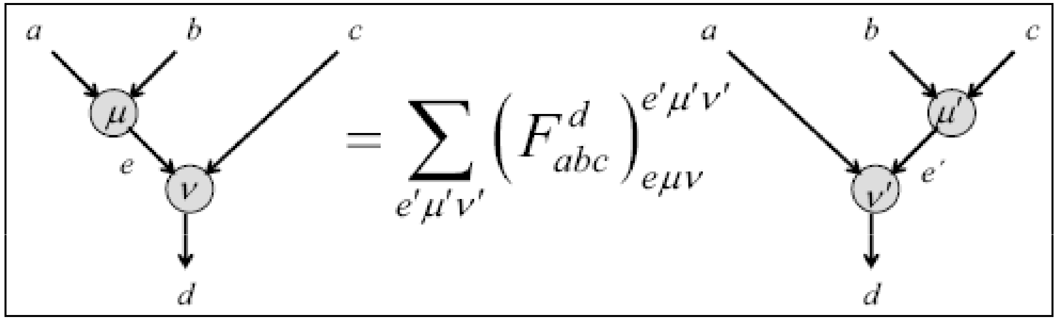



- orthogonality: $\langle ab; c'; \mu' \mid ab; c; \mu\rangle = \delta^{c}_{c'}\delta^{\mu}_{\mu'}$
- completeness: $\sum_{c,\mu} |ab;c;\mu\rangle\langle ab;c;\mu| = I_{ab}$
- trivial fusion: $1 \times a \longrightarrow a$

***non-abelian if:*** $\dim\left(\bigoplus_c V_{ab}^c\right) = \sum_c N_{ab}^c \geq 2$

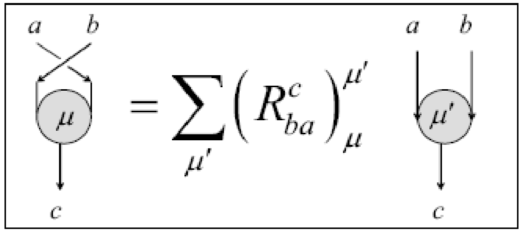
# Anyon Model – 2. Associativity: F-Matrix

- **3-particle fusion**: two ways to decompose topological Hilbert space
- **Charge conservation**: requires associativity




$$V^{d}_{abc} \cong \bigoplus_{e} V^{e}_{ab} \otimes V^{d}_{eb} \cong \bigoplus_{e'} V^{d}_{ae'} \otimes V^{e'}_{bc}$$

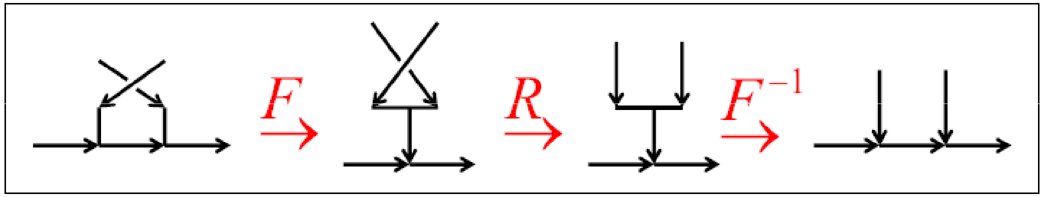
# Anyon Model – 3. Braiding: R-Matrix




- Isomorphic map from $\mathbf{V}^{c}_{ab}$ to $\mathbf{V}^{c}_{ba}$ : $R\!: \; |ba;c;\mu\rangle = \sum_{\mu'} |ab;c;\mu'\rangle \; (R^{c}_{ab})^{\mu'}_{\mu}$
- Monodromy operator $R^2$: $(R^{c}_{ab})^2 = \underbrace{diag}_{c}\left[e^{i(\theta_c - \theta_a - \theta_b)}\right]$

# Anyon Model – The Standard Basis

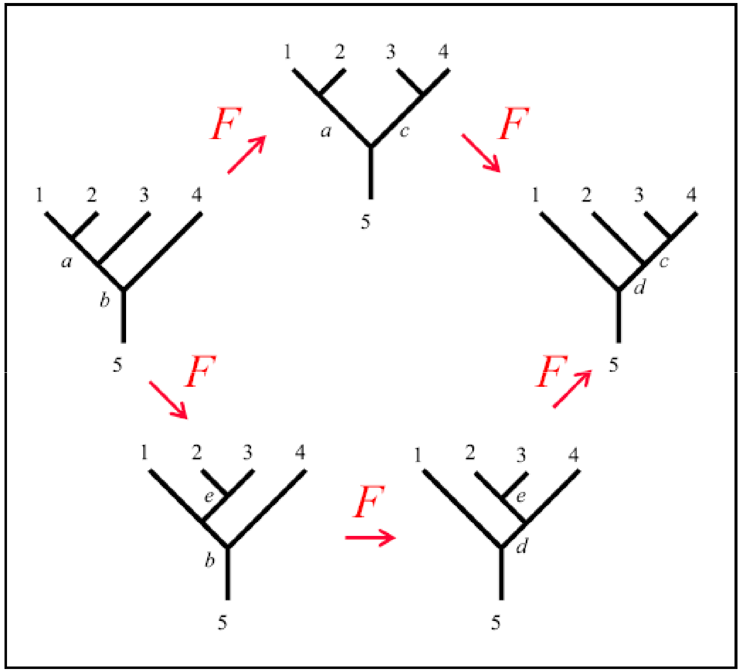
- Choose a standard basis
- B-matrices: transform diagonalized braid matrices to standard basis
- Coverage with B-matrices of SU(2) is key to universality of braiding operations

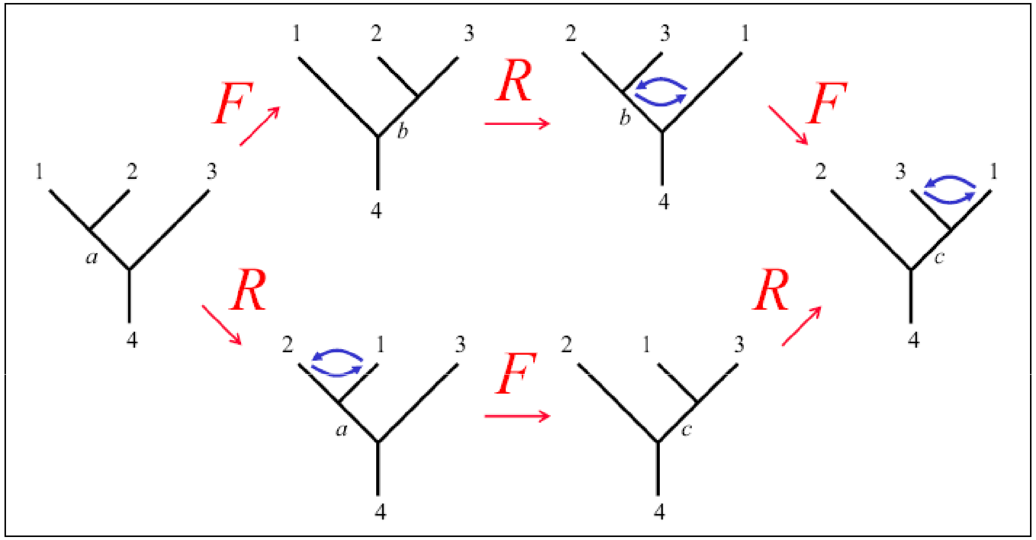



$$
\begin{aligned}
B|(ac)b \to d; e\rangle &= \sum_f B|a(cb) \to d; f\rangle \left(F^d_{acb}\right)^f_e \\
&= \sum_f |a(bc) \to d; f\rangle R^f_{bc} \left(F^d_{acb}\right)^f_e \\
&= \sum_{f,g} |(ab)c \to d; g\rangle \left[\left(F^{-1}\right)^d_{abc}\right]^g_f R^f_{bc} \left(F^d_{acb}\right)^f_e
\end{aligned}
$$

# Anyon Model – Consistency

***Moore-Seiberg*** Polynomial Equations




- Lowest-order consistency equations for fusion and braiding
- Pentagon Equation: $(F^5_{12c})^d_a (F^5_{a34})^c_b = (F^d_{234})^c_e (F^5_{1e4})^d_b (F^b_{123})^e_a$
- Hexagon Equation: $R^c_{13} (F^4_{213})^c_a R^a_{12} = \sum_b (F^4_{231})^c_b R^4_{1b} (F^4_{123})^b_a$

# Anyons – Where's the Real World (3)

## Thought-Experiment on the Manipulation of Fusion Channels

1. create a pair of fluxons from vacuum

2. create a pair of chargeons

3. move one chargeon around a fluxon $|a> \rightarrow D * |a> = |a'>$

4. look at total charge of pair $\langle a|a'\rangle^2 = \left|\frac{\chi^R(a)}{|R|}\right|^2$

**Result:**

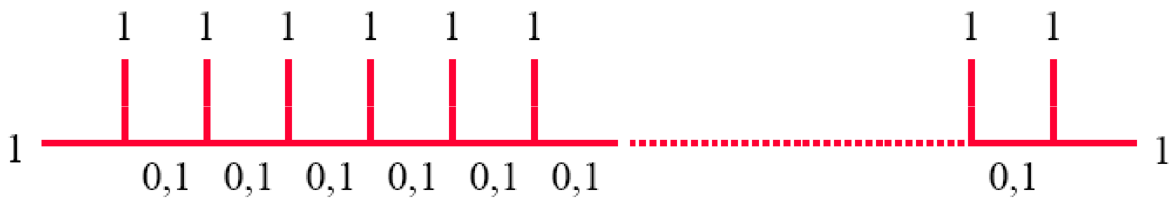
• charge ‚wanders' from chargeon to fluxon
• charges will not annihilate any more, but overall charge is preserved → fusion channel

# Playing by the rules – The Yang-Lee (Fibonacci) Model

# The Yang-Lee Model – Definition

- Simplest model
- Two labels ($\tau$ and 1)
- Single fusion rule: $\tau * \tau \rightarrow \tau + 1$
- Anyons created from vacuum: zero-charge constraint

**But what is the dimension of our Hilbert Space?**




**Constraint**: no two trivial labels in a row

→ Dimension of n-Anyon Hilbert Space: $n^{th}$ Fibonacci Number

$$F = \begin{bmatrix} \sigma & e^{i\phi}\sqrt{\sigma} \\ e^{-i\phi}\sqrt{\sigma} & -\sigma \end{bmatrix} \qquad R = \begin{bmatrix} e^{\frac{4}{5}\pi i} & 0 \\ 0 & -e^{\frac{2}{5}\pi i} \end{bmatrix}$$

- Braid group generators: $\sigma_1 \longmapsto B_1 = R\,, \quad \sigma_2 \longmapsto B_2 = F^{-1}RF$

# The Yang-Lee Model – Definition

- Now, solve ***Moore-Seiberg Equations***

$$F = \begin{bmatrix} \eta & e^{i\phi}\sqrt{\sigma} \\ e^{-i\phi}\sqrt{\sigma} & -\eta \end{bmatrix}$$

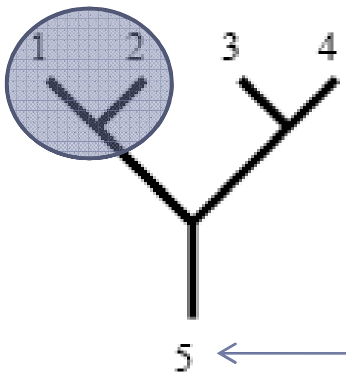
- eta: 1 – golden ratio
- phi: freely choosable phase

$$R = \begin{bmatrix} e^{\frac{4}{5}\pi i} & 0 \\ 0 & -e^{\frac{2}{5}\pi i} \end{bmatrix}$$

- Braid group generators: $\sigma_1 \longmapsto B_1 = R\,, \quad \sigma_2 \longmapsto B_2 = F^{-1}RF$

# The Yang-Lee Model – The Qubit

- Represent qubit by 4 anyons generated from vacuum
  - Total Charge: 1 (trivial charge)
  - Dimension of Hilbert Space: 2
- Standard basis: sequential fusion
- Where's our information? Really just in a 2-anyon pair




must be trivial particle

**Quantum Gates:** $\sigma_1 \mapsto B_1 = R\,, \quad \sigma_2 \mapsto B_2 = F^{-1}RF$

# The Yang-Lee Model – Computation

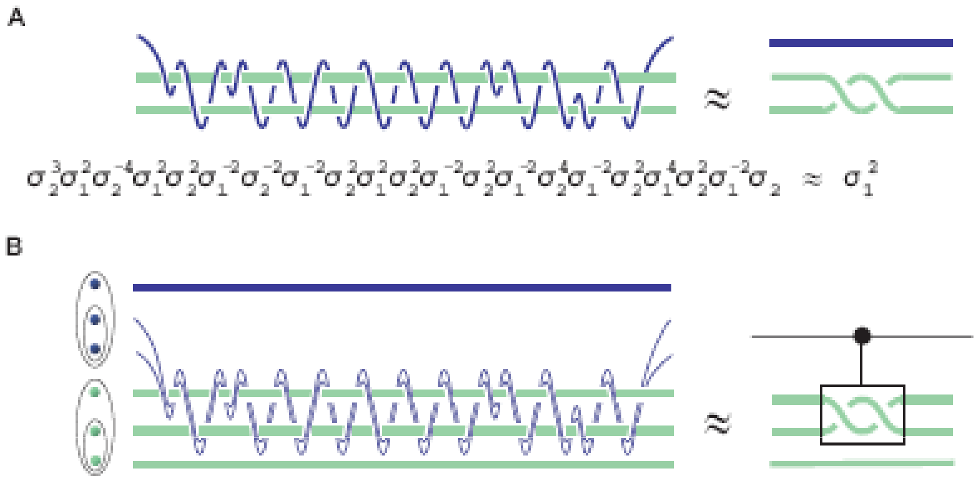



FIG. 16 Construction of a two qubit gate from a certain three particle problem. ime flows from left to right in this picture. In the top we construct a braid on three strands moving only the blue particle which has the same effect as interchanging the two green strands. Using this same braid (bottom), then constructs a controlled rotation gate. If the state of the upper (control) qubit is $|0\rangle$, i.e., the control pair is in state 1 then the braid has no effect on the Hilbert space (up to a phase). if, the upper (control) qubit is in the state $|1\rangle$ then the braid has the same effect as winding two of the particles in the lower qubit. Figure from Bonesteel et al., 2005

# The Yang-Lee Model – Accuracy

braiding operations dense in SU(2) --> arbitrary accuracy possible

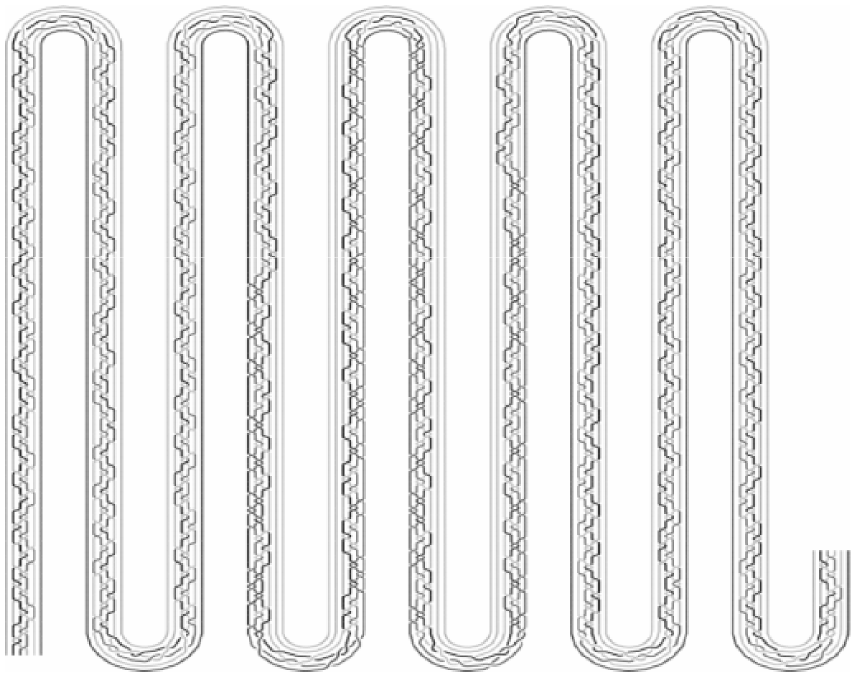


Figure 14.29: Improved version of the CNOT with an error of $\sim 10^{-4}$. The braid length used to achieve a ten times better accuracy grows by a factor of $\approx 4$. Picture adopted from Ref. [145].

# Errors in topological quantum computers

# What is to be gained?

1. The relevant information is not locally accessible, so no **local perturbation** is irrelevant!

2. During computation the system will stay inside the degenerate ground state space, so no **energy relaxation** is possible!

# Which errors are relevant?

- Assumed possible topological errors:

1. Braiding with thermaly activated parasitious anyons:
   - Probability of generation scales with $e^{-\Delta/kT}$

   where $\Delta$ is the energy gap from ground to first exited state!

2. Topological errors of displacement during braiding:
   - Probability of the creation of topological errors while moving anyons scales with $e^{-\alpha l}$

   where $l$ is the characteristic length scale of the system

# Links to quantum error correcting codes

# Redundancy and decomposition

- Idea of QECC:
  represent n qubits by a system S
  with higher dimensional Hilbertspace: $\mathcal{H}_S \cong \mathbb{C}^d \quad 2n \leq d$

- Assumption on errors gives decomposition:

$$\mathcal{H}_S \cong \bigoplus_J \underbrace{\mathbb{C}^{n_J}}_{\text{noisefree subspace}} \otimes \overbrace{\mathbb{C}^{d_J}}^{\text{noisy subspace}}$$

- Errors act on $\mathbb{C}^{d_J}$ and $\mathbb{C}^{n_J}$ stays error free!

- Encoding such that information is stored in $\mathbb{C}^{n_J}$

# Noiseless subspaces

## QECC

- Encoding in m qubits:

$$\mathcal{H}_S = \{|0\rangle, |1\rangle\}^{\otimes m} \cong \mathbb{C}^{2m}$$

- Noisefree subspace $\mathbb{C}^{n_J}$ is not directly accessible!

- Error prone operations for error correction.

## Topological coding

- Encoding in an system with topological degrees of freedom:

Arbitrarily big, but comes for free!

$$\mathcal{H}_S = \mathcal{H}_{topo} \otimes \mathcal{H}_{local}$$

- Assuming local errors $\mathcal{H}_{topo}$ is a noisefree subspace
- For anyons $\mathcal{H}_{topo}$ is directly accessible by braiding and fusion!

# Summary

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- **Generalised anyons**
  - Components of a model
  - Fusion channels
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- **Comments on errors in topological quantum computers**
  - Errors in topological quantum computers
  - Links to quantum error correcting codes
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# Conclusions

„It is not yet clear whether quantum error correction or topologically protected quantum computation will be the future standard of quantum computation but the beautiful topology involved gives the latter an „intrinsic coolness factor“

“...but wait a second, how exactly do you move this anyon?”

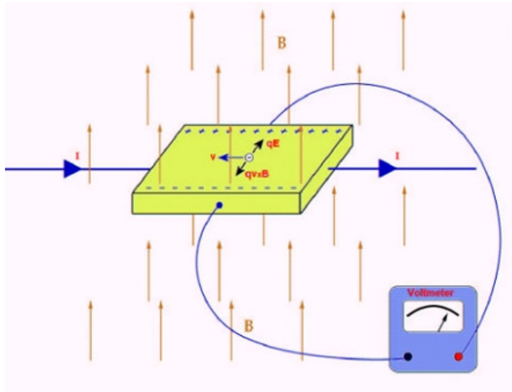
# End

# Backup slides

# Chasing the anyon -
# The fractional quantum hall effect

# Crash course on the FQHE (1)

- Classical Hall effect: $U_H = R_H \frac{IB}{d}$

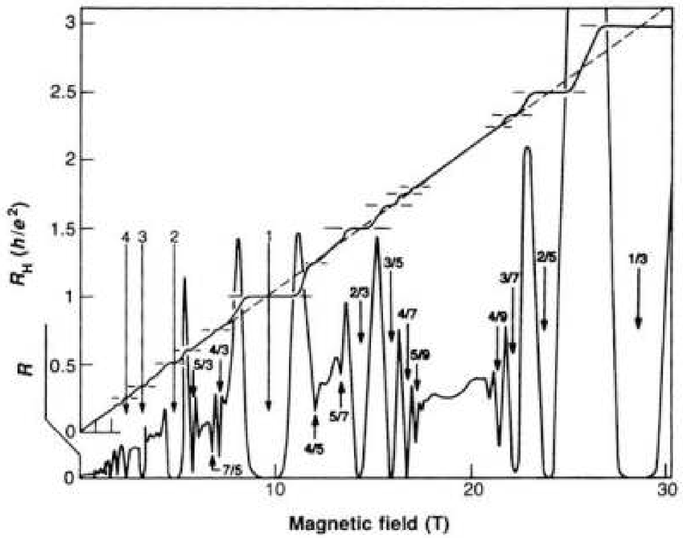


- For low temperature (~10mK) and strong fields (~5T) we see plateaus at

$$R_H = \frac{1}{m} \cdot \frac{h}{e^2} \quad m \in \mathbb{N}$$

- For even lower temps and stronger fields we see extra plateaus at

$$R_H = \frac{1}{\nu} \cdot \frac{h}{e^2} \quad \text{with} \quad \nu = \frac{p}{q} \quad p, q \in \mathbb{N}$$




Landau filling level $\nu = n_{electrons} \, {}^{hc}/_{eB}$

# Crash course on the FQHE (2)

- FQHE can be described by fractionally charged quasi-particle exitations for filling factors

$$\nu = \frac{q}{2q+1} \qquad \tilde{e} = \frac{e}{2q+1}$$

- As expected for fractional charge they exibit fractional statistics. A counter clockwise exchange results in a phase $e^{-i\pi\nu}$ on the wave function.

- These quasi-particles are expected to be anyons! Experimental proof is achieved (but doubted).

# From Fusion Channels to Physics

# Anyons – Where's the Real World (1)

Let's do a thought experiment:

- „superconductor"
- 2 particle quantities: ***charge*** and ***flux*** → „chargeons" and „fluxons"
- gauge-invariant theory (i.e. electrodynamics, chromodynamics)
- consider non-abelian finite group G:
  - flux takes value in G
  - charges are unitary irreducible representations of G
- assume no field-like interactions

# Anyons – Where's the Real World (2)

## Chargeons:

- let charges be described in a multi-dimensional space **R** (not a scalar anymore!!)
- establish standard basis: $|R, i\rangle \,, \quad i = 1, 2, \ldots |R|$
- Charge transported around a closed path enclosing a flux

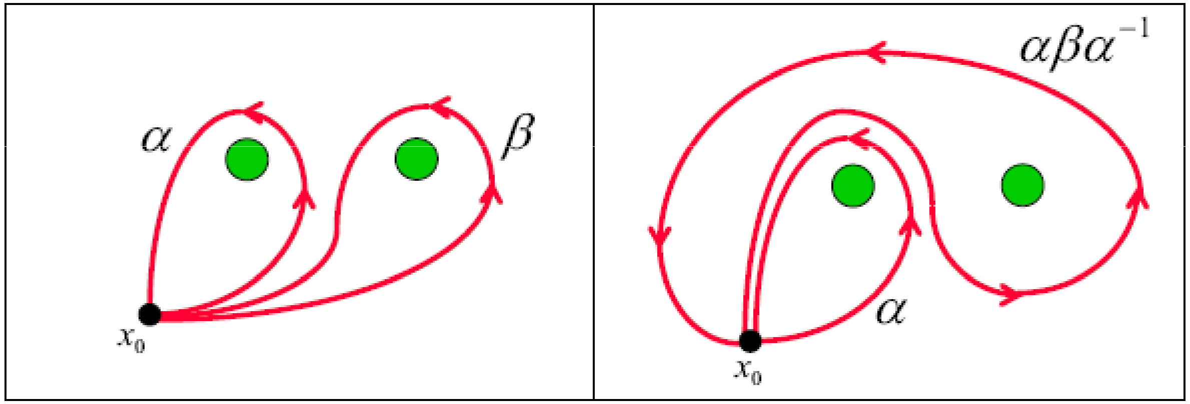
→ **non-scalar** Aharonov-Bohm effect

$$|R, j\rangle \mapsto \sum_{i=1}^{|R|} |R, i\rangle D_{ij}^{R}(a)$$

# Anyons – Where's the Real World (3)

## Fluxons: permutation and conjugacy class

1. assume two fluxons a (on the left) and b (on the right)
2. fluxons can be categorized in G by moving a charge around them and measuring its change
3. now, permutate a and b clockwise




Braiding operation: $R\colon |a,\, b\rangle \longrightarrow |b,\, b^{-1}ab\rangle$

Monodromy operator: $R^2\colon |a,\, b\rangle \longrightarrow |(ab)a(ab)^{-1}, (ab)b(ab)^{-1}\rangle$

**Note:** Permutating fluxes changes the flux element, but the conjugacy class remains invariant

# Anyons – Where's the Real World (3)

## Thought-Experiment on the Manipulation of Fusion Channels

1. create a pair of fluxons from vacuum
2. create a pair of chargeons $|0;\ R\rangle = \frac{1}{\sqrt{|R|}} \sum_i |R,\ i\rangle \otimes |\overline{R},\ i\rangle$
3. permutate a and b clockwise $|0;\ R\rangle \mapsto |0;\ R\rangle' = \frac{1}{\sqrt{|R|}} \sum_{i,j} |R,\ j\rangle \otimes |\overline{R},\ i\rangle D^R_{ji}(a)$
4. look at total charge of pair $\langle 0; R|0'; R'\rangle^2 = \left|\frac{\chi^R(a)}{|R|}\right|^2$

→ character of ***D***

**Result:**

- charge ‚wanders' from chargeon to fluxon
- charges will not annihilate any more, but overall charge is preserved → fusion channel