QSIT, 15.10.2007

- Coherent manipulations of QuBits:
  - i) single qubit rotations (on the Bloch sphere)
  - ii) conditional quantum dynamics
    i.e. CNOT gate
- Measurements

• QUANTUM Measurements

- Measurement postulate:

1.) Observables in a q-measurement are described by Hermitian operators:

$$\hat{A} = \sum_n a_n |n\rangle\langle n| = \hat{A}^{\dagger}$$

2.) Outcome of the measurement is a EV of $\hat{A}$ with probability:

$$\langle P_n\rangle = \langle \psi(t)|\hat{P}_n|\psi(t)\rangle$$

$\hat{P}_n$ ... projection operator $P_n = |n\rangle\langle n|$

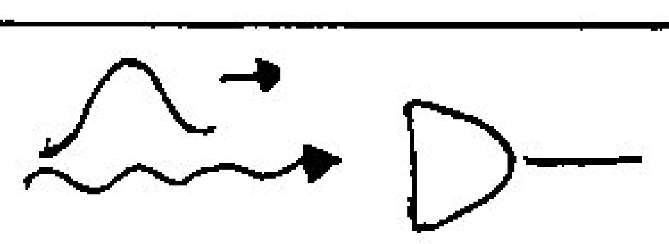
3.) the final state after the measurement is

$$|\psi'(t)\rangle = \frac{\hat{P}_n|\psi(t)\rangle}{\sqrt{p(n)}}$$

1.) - 3.) PAULI'S FIRST KIND Measurement
(projective or orthogonal measurement)

- PAULI'S 2^{nd} KIND Measurement ~~(1) (2), but not (3)~~

E.g.: Photodetection



$\hat{N}$ is measured, but final state is not $|n\rangle$, but a certain # of e^-s

- Measurements are in practice imprecise:

i.e. prepare $|\psi(t)\rangle = |n_1\rangle$, where $\hat{A}|n_1\rangle = a_{n_1}|n_1\rangle$

⟹ Generalisation of measurement postulate required for better description of real measurements.

•) GENERALIZED Measurement Postulate

1.) A measurement is described by the set of operators $\{\hat{M}_m\}$ (in general non-Hermitian and/or non-unitary)

2.) A possible measurement outcome m has probability

$$p(m) = \langle \psi(t) | M_m^\dagger M_m | \psi(t) \rangle$$

3.) Post-measurement state:

$$|\psi'(t)\rangle = \frac{\hat{M}_m |\psi(t)\rangle}{\sqrt{p(m)}}$$

4.) Condition:

$$\sum_m M_m^\dagger M_m = \mathbb{1} \quad \text{since} \quad \sum_m p(m) = 1$$

→ every possible outcome m has a corresponding operator $\hat{M}_m$.

- Application: photo detection

possible outcomes photon # m

Measurement: $m \Leftrightarrow M_m = |0\rangle\langle m|$

$$p(m) = \langle \psi | m \rangle \langle 0 | 0 \rangle \langle m | \psi \rangle = |\langle \psi | m \rangle|^2$$

post-measurement state

$$|\psi'\rangle = \frac{M_m |\psi\rangle}{\sqrt{p(m)}} = |0\rangle$$

•) NEUMARK'S THEOREM

Generalised measurement can be implemented by considering unitary evolution on an extended system + meter

Gen. measurement of a system = unitary evolution of an extended system + meter followed by a projective meas. that obeys the standard meas. postulate

$$H_{INT} = \sum_i \underbrace{\hat{A}_i}_{\text{system}} \otimes \underbrace{\hat{B}_i}_{\text{meter}}$$

b.) initial state $|\psi_2\rangle$

$$\langle\psi_2|\hat{E}_2|\psi_2\rangle = \emptyset \rightarrow \text{state was } |\psi_1\rangle$$

$\langle\psi_2|\hat{E}_1|\psi_2\rangle \approx 30\%$: ! this could not have been achieved with proj. measurement!

$\langle\psi_1|E_3|\psi_1\rangle = \langle\psi_2|E_3|\psi_2\rangle = \frac{1}{\sqrt{2}} \rightarrow$ no information gained

Note: coeff. of $\hat{E}_i$ chosen such that

$\langle\psi_1|E_3|\psi_1\rangle = \langle\psi_2|E_3|\psi_2\rangle$ so that for E_3 no information can be gained

- Example: NEUMARK'S THEOREM

Any POVM can be described as a projective measurement on a larger Hilbert space.

→ consider POVM:

non-projective since $P_i^2 \neq P_i$!

$$\begin{cases} \hat{E}_1 = |\uparrow_z\rangle\langle\uparrow_z|\frac{1}{2} & \hat{E}_3 = |\uparrow_x\rangle\langle\uparrow_x|\frac{1}{2} \\ \hat{E}_2 = |\downarrow_z\rangle\langle\downarrow_z|\frac{1}{2} & \hat{E}_4 = |\downarrow_x\rangle\langle\downarrow_x|\frac{1}{2} \end{cases}$$

$$E_3 = \frac{1}{2}\,\frac{|\uparrow_z\rangle + |\downarrow_z\rangle}{\sqrt{2}}\,\frac{\langle\uparrow_z| + \langle\downarrow_z|}{\sqrt{2}}$$

→ add an auxiliary qubit to the system, i.e. enlarge Hilbert space. ("ancilla qubit")

$$\rightarrow \mathcal{H} = S^2 \otimes S^2 = \mathcal{H}_A \otimes \mathcal{H}_B$$

→ find a state in $\mathcal{H}_B$ and a set of projectors $\{P_i\}$ describing a projective measurement on $\mathcal{H}$ such that

$$_A\langle\psi|\,_B\langle\phi|\hat{P}_i|\phi\rangle_B|\psi_A\rangle = {}_A\langle\psi|\hat{E}_i|\psi\rangle_A$$

i.e. both ways should give the same statistics.

$$P_1 = |\uparrow_z\uparrow_z\rangle\langle\uparrow_z\uparrow_z| \qquad P_2 = |\downarrow_z\uparrow_z\rangle\langle\uparrow_z\downarrow_z|$$

$$P_3 = |\uparrow_x\downarrow_z\rangle\langle\uparrow_x\downarrow_z| \qquad P_4 = |\downarrow_x\downarrow_z\rangle\langle\downarrow_x\downarrow_z|$$

i.e. whenever measuring with $\hat{P}_1$, $\hat{P}_2$ the ancilla qubit ~~along~~ ~~z~~ is measured in z-direction, z-basis is selected for choose:

$$|\phi\rangle_B = |\uparrow_x\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow_z\rangle + |\downarrow_z\rangle\right)$$

QSIT lecture, 29.10.2007

- QUANTUM ERROR CORRECTION (Shor, Steane) (QEC)

$$\hat{\rho}(t) = \sum_i M_i \hat{\rho}(0) M_i^\dagger \qquad \sum_i M_i^\dagger M_i = \mathbb{1}$$

information leakage to environment can be compensated

- CLASSICAL ERROR CORRECTION (CEC)

  - bits in channel with error

$$\{0,1\} \xrightarrow{\text{Alice}} \boxed{\text{channel}} \xrightarrow{\text{Bob}}$$

only one type of error: (bit-flip error)

$$\left.\begin{array}{l} 0 \to 1 \\ 1 \to 0 \end{array}\right\} \text{with error prob. } p$$

-> correction by means of redundancy: send multiple pulses $0 \to 000$
$1 \to 111$

-> measure each pulse

- if $p \ll 1$: $110 \to 111$ "MAJORITY VOTING"
  110 is taken as 111 since it is much more likely that only one bit has flipped than two
- with error correction:
  - prob that two bits flip: $P = 3p^2(1-p)$
  - prob that error correction fails:

$$P_{coded} = \underbrace{3p^2(1-p)}_{2\text{ bits flip}} + \underbrace{p^3}_{3\text{ bits flip}} < p \to \text{solve}$$

-> Error correction works if $p < \frac{1}{2}$!
-> If $p > \frac{1}{2}$: just invert majority vote and it works again!

=> Thus: classical error correction fails only if $p = \frac{1}{2}$! ($\hat{=}$ complete randomness)

- What is used for CEC: Problems for QEC

1) We copied the initial state $0 \to 000$
   → QEC: no-cloning theorem !
2) A quantum measurement necessarily disturbs / changes state
3) Unlike classical inf. processing; in QIP a continuum of errors is given:
   - phase-flip Z
   - bit-flips X
   - superpositions of both $Y = iXZ$
   - $M = X + Z$ also possible
   ⋮

- QEC: Start out with bit-flip correction

$|\psi\rangle$: with prob p $X|\psi\rangle$
with $(1-p)$ $|\psi\rangle$

Kraus operators: $M_0 = \sqrt{1-p}\,\mathbb{1}$ $M_1 = \sqrt{p}\,X$

$\Rightarrow$ $|0_L\rangle \equiv |000\rangle$
$|1_L\rangle \equiv |111\rangle$

arbitrary state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

'logical state': $|\psi_L\rangle = \alpha|0_L\rangle + \beta|1_L\rangle$

- 3 errors possible: first, second or third bit in $|0_L\rangle$ and (X_1, X_2, X_3) $|1_L\rangle$, respectively, can flip with prob. p

$$|\tilde{\psi}_L\rangle \xrightarrow[\text{error channel}]{} \alpha|100\rangle + \beta|011\rangle \qquad \boxed{p \ll 1}$$

-> Measure parity $Z_1 Z_2$ and $Z_2 Z_3$ instead of Z_1

$Z_1 Z_2$: $+1$ -> both qubits in same state $|00\rangle$ or $|11\rangle$
-1 -> different states

$$\left.\begin{array}{l} Z_1 Z_2 |\tilde{\Psi}_L\rangle \rightarrow -1 \\ Z_2 Z_3 |\tilde{\Psi}_L\rangle \rightarrow +1 \end{array}\right\}$$ Thus: qubit one has flipped!

After parity measurement: if error has occurred, it can be corrected by:

$$X_1 \underset{\substack{\uparrow \\ \text{state after} \\ Z_1 Z_2 \text{ measurement}}}{|\tilde{\Psi}_L\rangle} = |\Psi_L\rangle$$

– let all qubits go through error-channel (A, B, C)

$$\hat{\rho}(t) = \sum_i M_i \hat{\rho}(0) M_i$$

$$M_0 = \sqrt{(1-p)^3}\; \mathbb{1}_A \otimes \mathbb{1}_B \otimes \mathbb{1}_C$$
$$M_1 = \sqrt{p(1-p)^2}\; \hat{X}_A$$
$$M_2 = \sqrt{p(1-p)^2}\; \hat{X}_B \qquad M_4 = \sqrt{p^3}\, X_A X_B X_C$$
$$M_3 = \sqrt{p(1-p)^2}\; \hat{X}_C \qquad \sum_{i=0}^{3} M_i^\dagger M_i \approx \mathbb{1}$$
$$\vdots$$

•) assume that $Z_A Z_B = 1$ and $Z_B Z_C = 1$
→ either there has been no error or maximum error, i.e. all bits flipped!, i.e.

$$\hat{\rho}(t) = \hat{\rho}(0) \quad \text{or} \quad \hat{\rho}(t) = X_A X_B X_C\, \hat{\rho}(0)\, X_A X_B X_C$$
with prob. p^3

– Fidelity: measurement of corruption

$$F := \langle \psi | \rho(t) | \psi \rangle$$

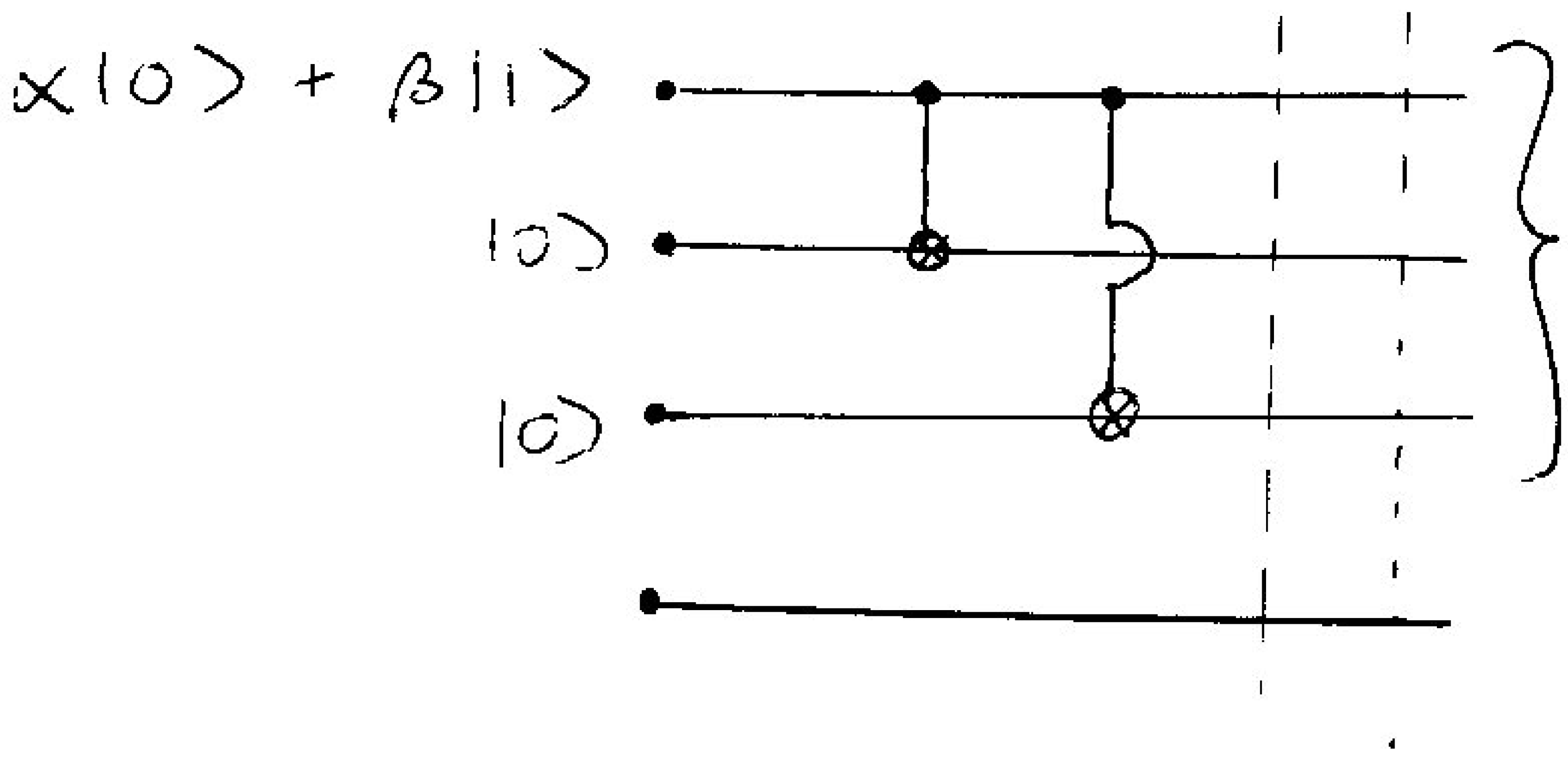
no error correction (1 qubit) $F = 1 - p$

→ min. fidelity after error-correction + detection:
$$F = 1 - 3p^2(1-p) - p^3$$

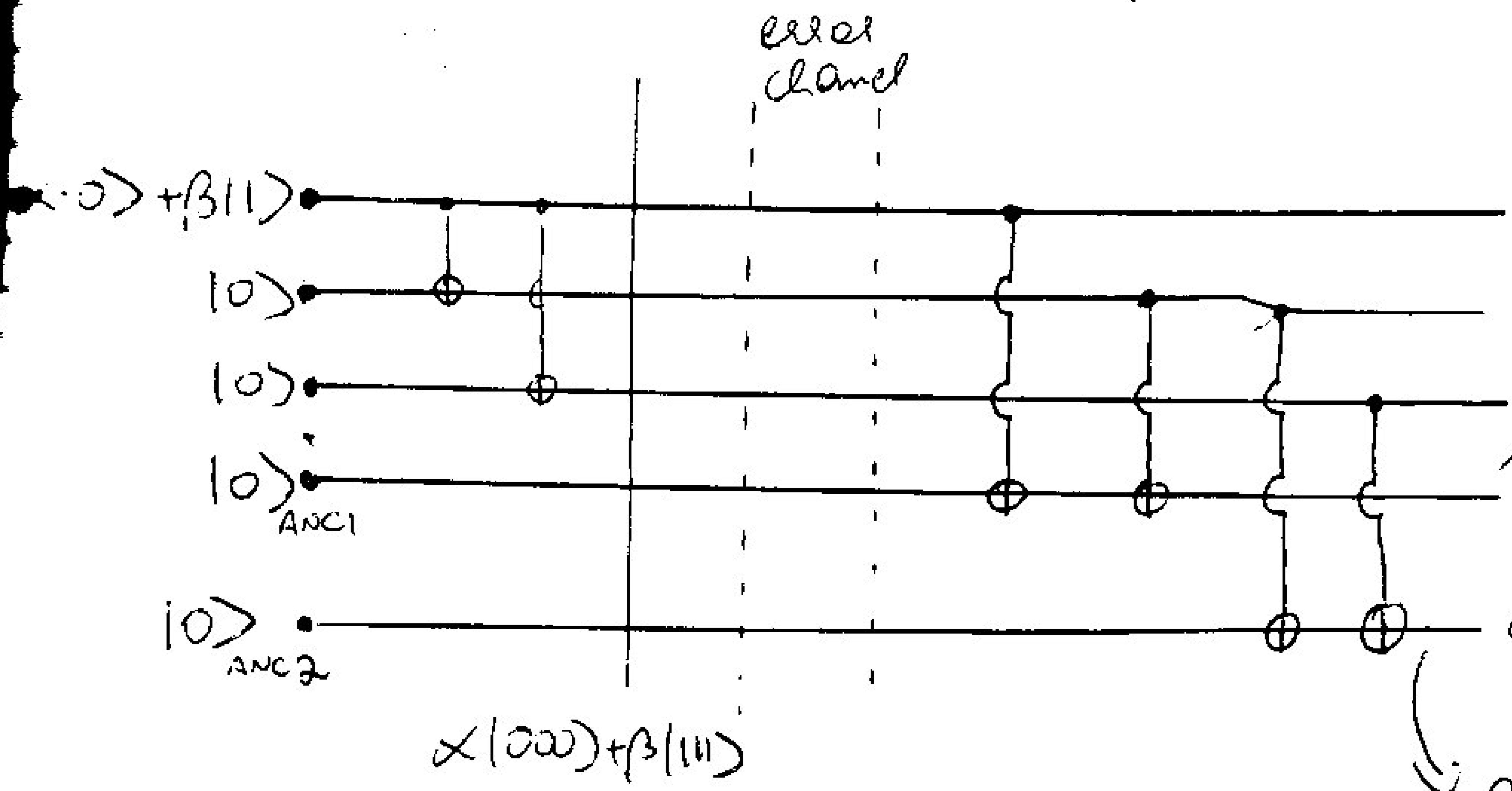
•) assume that $Z_A Z_B = -1$, $Z_B Z_C = +1$

- ENCODING of QEC

$\alpha|0\rangle + \beta|1\rangle \rightarrow \alpha|000\rangle + \beta|111\rangle$ by means of CNOTs



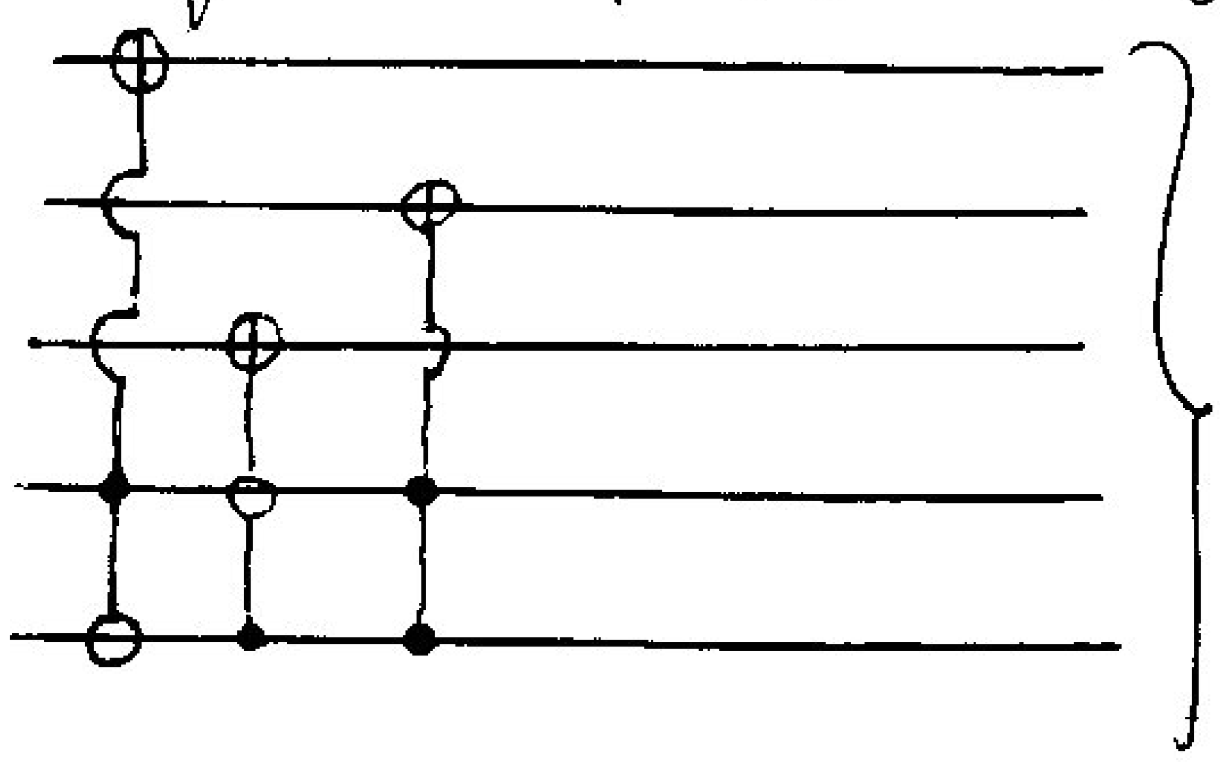
$\alpha|000\rangle + \beta|111\rangle$



~ projective measurement + on the ancilla 1 $\hat{=} Z_1 Z_2$ measurement

proj. measurement ancilla 2 $\hat{=} Z_2 \cdot Z_3$

Now if error has occured, apply TOFFOLI-Gate: flip the 3rd qubit if both 1 and 2 are the same.



$(\alpha|000\rangle + \beta|111\rangle) \otimes |1\rangle_1 |0\rangle_2$

PHASE-FLIP ERROR $\hat{Z}$

$\alpha|0\rangle + \beta|1\rangle \rightarrow \alpha|0\rangle - \beta|1\rangle$

QEC: use <u>duality</u> b.w. $\hat{X}$ and $\hat{Z}$

$\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) := |+\rangle$

$\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) := |-\rangle$

$\hat{Z}|+\rangle = |-\rangle \quad \hat{Z}|-\rangle = |+\rangle$

→ action $\hat{Z}$ on $|\pm\rangle$ is analogous to action of $\hat{X}$ on $|0\rangle/|1\rangle$!

$$\begin{array}{ccc} \alpha|0\rangle + \beta|1\rangle & \rightarrow & \alpha|+\rangle + \beta|-\rangle \\ \downarrow & & \\ \alpha|000\rangle + \beta|111\rangle & \rightarrow & \alpha|+++\rangle + \beta|---\rangle \end{array} \Bigg\} \text{ ENCODING}$$

→ can be achieved by applying H to first three qubit states before error channel

$$Z_1 Z_2 \rightarrow X_1 X_2$$

$$X_1 X_2 |+++\rangle = +1\, |+++\rangle$$
$$X_1 X_2 |+-+\rangle = -1\, |+-+\rangle$$

$$H_1 H_2 H_3 Z_1 Z_2 H_1 H_2 H_3 \hat{=} X_1 X_2$$

$$\left(\text{use} \quad \begin{array}{l} HZH = X \\ HXH = Z \end{array}\right)$$

i.e. phase flips have been converted into bit flips and then corrected.

- COMBINATION: SHOR'S 9-QUBIT CODE

$$|0_L\rangle = (|\overset{123}{000}\rangle + |111\rangle) \otimes (|\overset{456}{000}\rangle + |111\rangle) \otimes (|\overset{789}{000}\rangle + |111\rangle) \frac{1}{2\sqrt{2}}$$

$(|000\rangle + |111\rangle)$ corrects for bit flips

$(|000\rangle + |111\rangle)^{3\otimes}$ corrects for phase flips

$$|1_L\rangle = (|000\rangle - |111\rangle) \otimes (|000\rangle - |111\rangle) \otimes (|000\rangle - |111\rangle) \frac{1}{2\sqrt{2}}$$

- Bit flip on qubit 5:

$$\begin{array}{cccccc} Z_1 Z_2, & Z_2 Z_3, & Z_4 Z_5, & Z_5 Z_6, & Z_7 Z_8, & Z_8 Z_9 \\ 1 & 1 & \underbrace{-1 \quad\quad\quad -1} & & 1 & 1 \end{array}$$

tells you that flip has occurred in block 2 and that it was qubit 5

- phase flip on qubit 4:

$$\left.\begin{array}{l} \underbrace{X_1 X_2 X_3 X_4 X_5 X_6}_{-1} \cancel{X_7 X_8 X_9} \quad \text{phase of first block} \\ \text{and} \quad \underbrace{X_4 X_5 X_6 X_7 X_8 X_9}_{-1} \end{array}\right\} \text{apply either } Z_4, Z_5 \text{ or } Z_6$$

- bit and phase flip of qubit 6:

Any error can be corrected since the operators

$$\mathbb{1}, X, Y, Z$$

form a basis for all 2×2 matrices

$$M_i = \alpha \mathbb{1} + \beta \hat{X} + \gamma \hat{Y} + \delta \hat{Z}$$

i.e. by means of measurement M_i, the error is projected onto one of the basic errors $\hat{X}, \hat{Y}, \hat{Z}$ or $\mathbb{1}$.

- QSIT Exercises, 29.10.2007

"QUANTUM EVOLUTIONS"

- Time evolution of closed systems: SE

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = H |\psi(t)\rangle$$

→ unitary dynamics $U(t, t_0 = \emptyset) = \exp[-\frac{i}{\hbar} H t]$

- Liouville v. Neumann

$$\frac{\partial}{\partial t} \rho(t) = \frac{1}{i\hbar} [H, \rho]$$

→ dynamics: $\rho(t) = U(t, t_0)\, \rho(t_0)\, U^{\dagger}(t, t_0)$

Two-level system: $|g\rangle, |e\rangle$

let $t \to \infty$: $|e\rangle \to |g\rangle$

$|g\rangle \to |g\rangle$

- General time evolution: (superoperator evolution)

Let $\mathcal{H}_A$ be system A's Hilbert space and $\mathcal{H}_B$ the environment's Hilbert space.

→ assume initial state: $\rho_A \otimes |0\rangle_B\, {}_B\langle 0|$

i.e. it has been assumed that $t=\emptyset$, there is no interaction, and that's why the density operator can be written as a tensor product

→ time evolution:

$$U_{AB}\left(\rho_A \otimes |0\rangle_B\, {}_B\langle 0|\right) U_{AB}^\dagger$$

→ to observe time evolution of A, B has to be traced out:

$$\underline{\rho_A'} = Tr_B\left(U_{AB}\left(\rho_A \otimes |0\rangle_B\, {}_B\langle 0|\right) U_{AB}^\dagger\right)$$

$$= \sum_\mu {}_B\langle \mu| U_{AB}\left(\rho_A \otimes |0\rangle_B\, {}_B\langle 0|\right) U_{AB}^\dagger |\mu\rangle_B$$

$$= \sum_\mu {}_B\langle \mu| U_{AB} |0\rangle_B\, \rho_A\, {}_B\langle 0| U_{AB}^\dagger |\mu\rangle_B$$

$$= \underline{\sum_\mu M_\mu \rho_A M_\mu^\dagger}$$

Thus: The time evolution given by the transformation M_μ of system A depends on the state of the environment!

$$\rho_A' = \sum_\mu \rho_\mu \quad \text{state of A given B is in } |\mu\rangle$$

· for $\mu=1$, i.e. only one state

$$\rho_A' = \underbrace{M_\mu}_{\text{unitary}} \rho_A M_\mu^\dagger$$

•) Superoperator:

$$\Lambda : \rho_A \rightarrow \Lambda(\rho_A) = \underbrace{\sum_\mu M_\mu \rho_A M_\mu^\dagger}_{\text{KRAUS representation}}$$

(1) preserves Hermiticity

$$[\Lambda(\rho_A)]^\dagger = \Lambda(\rho_A)$$

(2) preserves trace

$$\mathrm{Tr}\,\Lambda(\rho_A) = \mathrm{Tr}\left(\sum M_\mu \rho_A M_\mu^\dagger\right) = \mathrm{tr}\,\rho_A = 1$$

(3) $\Lambda(\rho_A)$ is positive:

$$\langle\psi|\Lambda(\rho_A)|\psi\rangle = \sum_\mu \underbrace{\langle\psi|M_\mu}_{\geq 0} \rho_A \underbrace{M_\mu^\dagger|\psi\rangle}_{\geq 0} \geq 0$$

In more general terms, certain properties can be assumed a time evolution superoperator has to have. From this it can be concluded that any superoperator meeting these requirements has a Kraus representation.

PROPERTIES: (1) – (3)

(4) Linearity

$$\Lambda(\lambda\rho_1 + (\lambda-1)\rho_2) = \lambda\Lambda(\rho_1) + (\lambda-1)\Lambda(\rho_2)$$

(3) COMPLETELY POSITIVE

$$\Lambda : \rho_A \rightarrow \Lambda(\rho_A) = \rho_A'$$

Arbitrary Hilbert space $\mathcal{H}_C$ and consider $\Lambda \otimes \mathbb{1}_C$.
Λ is completely positive if $\Lambda \otimes \mathbb{1}_C$ is positive for all $\mathcal{H}_C$.

- Example: positivity but not complete positivity
TRANSPOSITION

$T : \rho \to T(\rho) = \rho^T$

•) T is positive:

$$\langle \psi | \rho^T | \psi \rangle \geq 0$$

•) Is T completely positive:
add a second qubit $\hat{=} \mathcal{H}_C$

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B = \{ |00\rangle, |01\rangle, |10\rangle, |11\rangle \}$$

→ Consider $T_A \otimes \mathbb{1}_B$

→ take $|\phi^+\rangle_{AB} = \frac{1}{\sqrt{2}} \left(|00\rangle + |11\rangle \right)$

$$|\phi^+\rangle_{AB} \, {}_{AB}\langle \phi^+| = \frac{1}{2} \left(|00\rangle\langle 00| + |11\rangle\langle 11| \right)$$
$$\left(|00\rangle\langle 11| + |11\rangle\langle 00| \right)$$

Apply $(T_A \otimes \mathbb{1}_B)$:

$$\left(|\phi^+\rangle_{AB} \, {}_{AB}\langle \phi^+| \right)' = \frac{1}{2} \left(|00\rangle\langle 00| + |11\rangle\langle 11| \right)$$
$$\left(|10\rangle\langle 01| + |01\rangle\langle 10| \right)$$

$$= \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{EV: } \frac{1}{2}(1, 1, 1, -1)$$

•) Criterion for entanglement (PEREZ-HORODECKI CRITERION)

every ~~entangle~~ state that is an entangled state has negative EVs under partial transpose.

- def. entanglement

state $|\psi\rangle$ is entangled if it cannot be written as $|\alpha\rangle_A \otimes |\beta\rangle_B$, more general:

$$|\psi\rangle = \sum_{i=1}^{N} \lambda_i |\alpha_i\rangle_A \otimes |\beta_i\rangle_B$$

for $N = 1$: no entanglement

for $N > 1$: entangled

$$\hat{U} = \exp\left[-\frac{i}{\hbar} H_{INT} t\right] \neq \hat{U}_A \otimes \hat{U}_B$$

Assume at $t = \emptyset$:

$$|\phi\rangle = |\psi\rangle_A \otimes |0\rangle_B$$

at $t = t_1$:

$$\hat{U}(t_1)|\phi\rangle = \sum_m \hat{M}_m |\psi\rangle_A \otimes |m\rangle$$

$|m\rangle$ .. orthonormal basis for meter B.

$$\text{or} \quad = \sum_n (\hat{N}_n |\psi\rangle_A) \otimes |n\rangle$$

- Unitary evolution preserves inner prod.s

-> after unitary evolution, apply projective measurement:

$$P_m = \mathbb{1}_A \otimes |m\rangle_B {}_B\langle m|$$

$$p(m) = \langle\psi| U^\dagger(t_1) P_m U(t_1) |\psi\rangle$$
$$= \langle\psi| M_m^\dagger M_m |\psi\rangle$$

- final state:

$$|\psi'\rangle = P_m \sum_{m'} M'_m |\psi\rangle \otimes |m'\rangle$$

$$= \underbrace{\frac{M_m|\psi\rangle}{\sqrt{p(m)}}}_{|\psi'\rangle_A} \otimes \underbrace{|m\rangle}_{|\psi'\rangle_B}$$

•) If $M_m = |m\rangle_A \langle m|$ -> generalised measurement is equivalent to a direct projective meas. on system A.

•) <u>POVM - Positive operator valued measurement</u>

- Scenario: implement a generalised measurement; we are interested in the outcome, but not in the post-measurement state

outcome m <-> $M^\dagger_m M_m := \hat{E}_m$ ... positive operator

$$p(m) = \langle\psi| M_m^\dagger M_m |\psi\rangle \qquad \sum_m M^\dagger_m M_m = \sum_m \hat{E}_m = \mathbb{1}$$

$\lambda \sim 300$ nm diode lasers

P.H.C.: necessary cond. for separability of a state

ρ is separable, i.e. not entangled, if the partial transpos of ρ, i.e. $(T_A \otimes 1\!\!1_B)\rho$, has only positive EVs.

QSIT, 19.11.2007

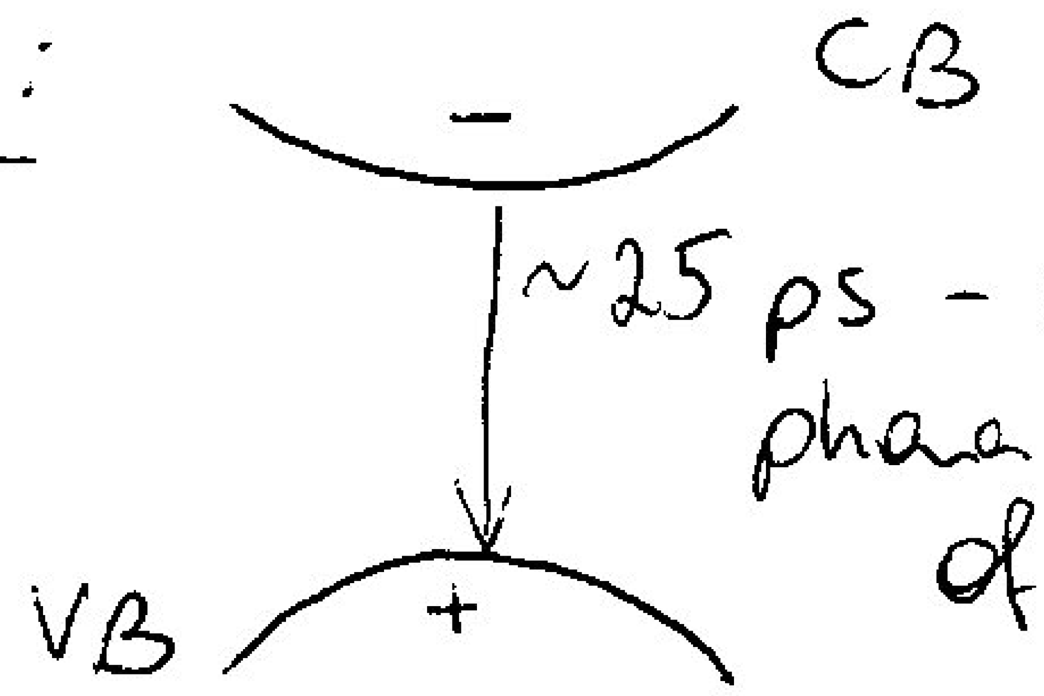
- Special date: Wednesday 10:45 – 12:30 HPT G18 (Seminar room)
  Nov. 28th

- SPIN QUBITS
  Semiconductor implementations
  questionable qubit implementation
  QUESTIONS:

  A) Why SPIN QUBITS
  - best isolation, best controllability: single trapped ions
    -> best candidate for systems up to 10 qubits
    (fidelity CNOT-gate: 99 %)
    -> problem: 10 000 qubits min. to do stuff you couldn't do with classical computers
    -> SCALABILITY for large ion numbers (high "OVERHEAD")
  -> for optical manipulation of ions, they have to be seperated by $d > \lambda \sim 0.5\,\mu m$ since they are all identical (in principle)
  -> Thus: size of computer would scale $\propto N_{QUBITS} \times \lambda$
  -> in principle, one could chose λ to be smaller, i.e. ions to have higher transition frequencies, but currently lasers are only available until $\lambda \sim 300$ nm (diode lasers)
  -> A: SCALABILITY

Semiconductor qubits: very low coherence times T_1
-> fs-lasers would be needed
-> thus: spin qubits are used, since spin degrees of freedom are remarkably well protected from other degrees of freedom.

- example:

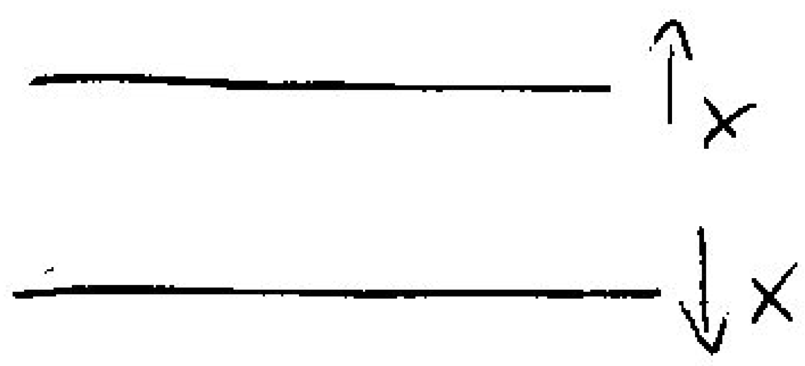


~ 25 ps – radiative decay
phonon induced flip/decoherence of holes ~ 60ps

→ very strong coupling to lattice

→ mid 90s: CB states are formed from anti-bonding s-orbitals – weak spin orbit coupling
→ spin is well-conserved!
VB states: bonding p orbitals – strong spin-orbit int.

→ decay time in CB of spin: $\tau \sim 1\,\mu s$ measured (GaAs)
- generate B_x – field
at $t = \emptyset$: e^- spins in $|\uparrow_z\rangle$ so that

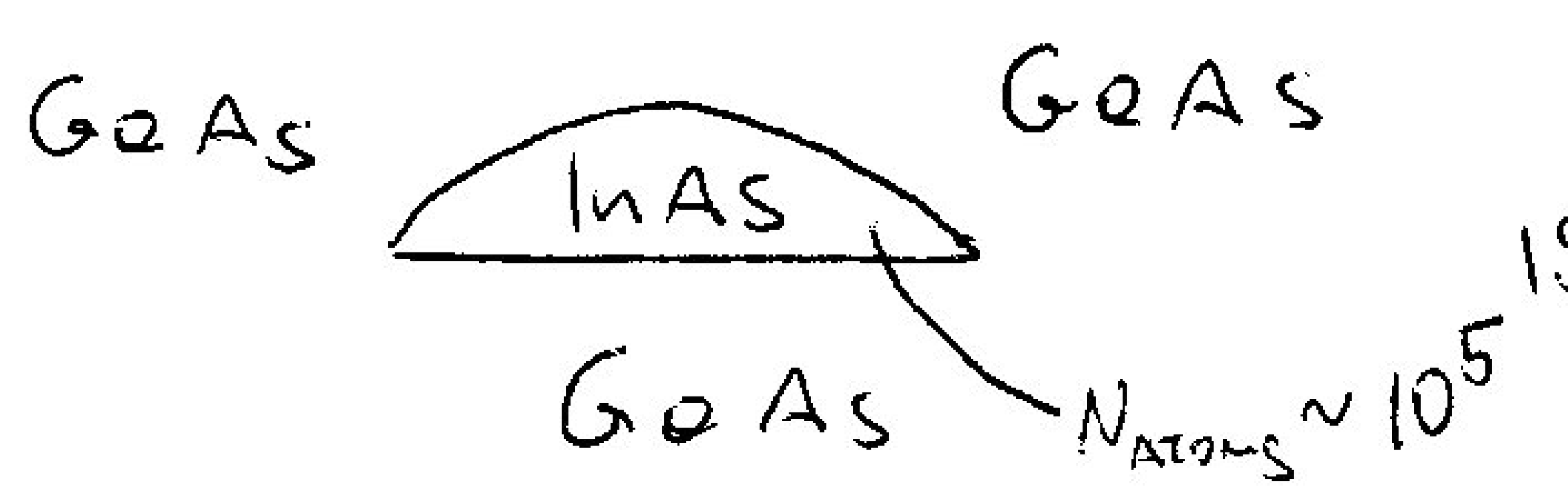


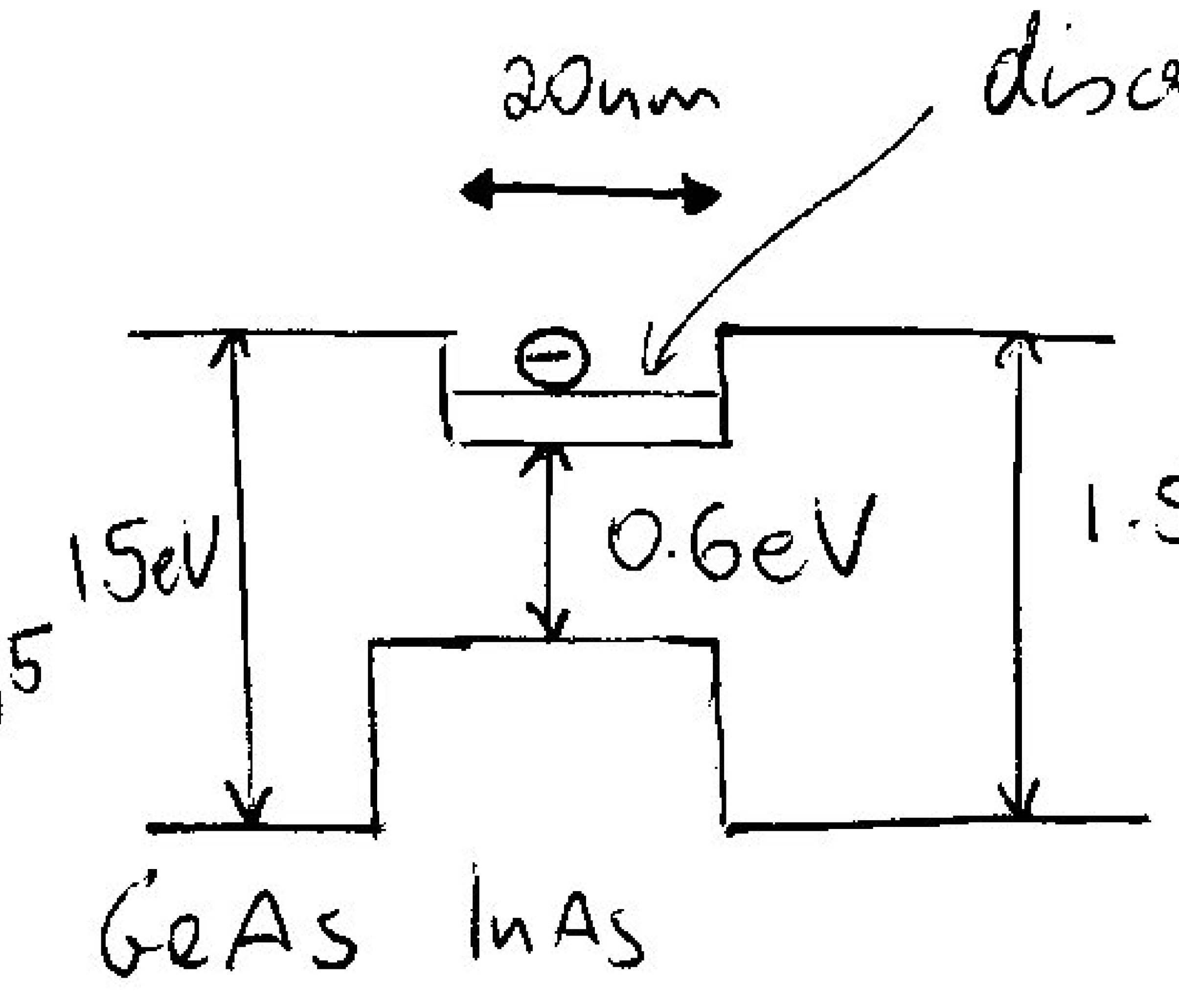
→ oscillations
→ measure period + decay time

in Si: even $\tau \sim 1$ ms measured
because even weaker spin-orbit coupling
because Si-atoms are lighter than GaAs

(experiments have been done in bulk material, i.e. no isolated qubits)

- QUANTUM DOTS (QD)
confined, isolated single spins
- small QD: problem of hyperfine interactions ≙ new decoherence mechanism
- Hyperfine Interaction





→ motion quantised in all spatial dimensions problem: hyperfine coupling of confined e^-'s spin and atomic spins in the QD

$A_H \sim 100\ \mu eV$

→ nuclear spins create an effective magnetic field seen by the e^- spin:

$\langle B_{eff} \rangle = \emptyset$ but $\Delta B_{eff} = B_{rms} = A_{HY} / \sqrt{N}$

(poisson distributed)

$B_{rms} \sim 10$ mT with unknown direction

→ ERROR; decoherence times ~ 3 ns

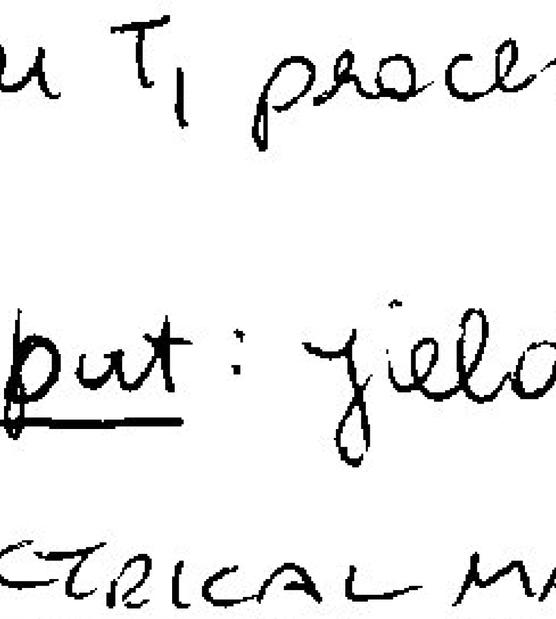
↳ Thus: Spin-Echo used to erase the accumulated, unknown phases → coherence times extended to ~ 3 ms

→ better solution: use Si since it has ~80% isotopes with zero nuclear spin.

- other T_1 process: spin flip co-tunnelling of e^-'s into the gap

→ but: yields highly interesting physics (Kondo-Effect)

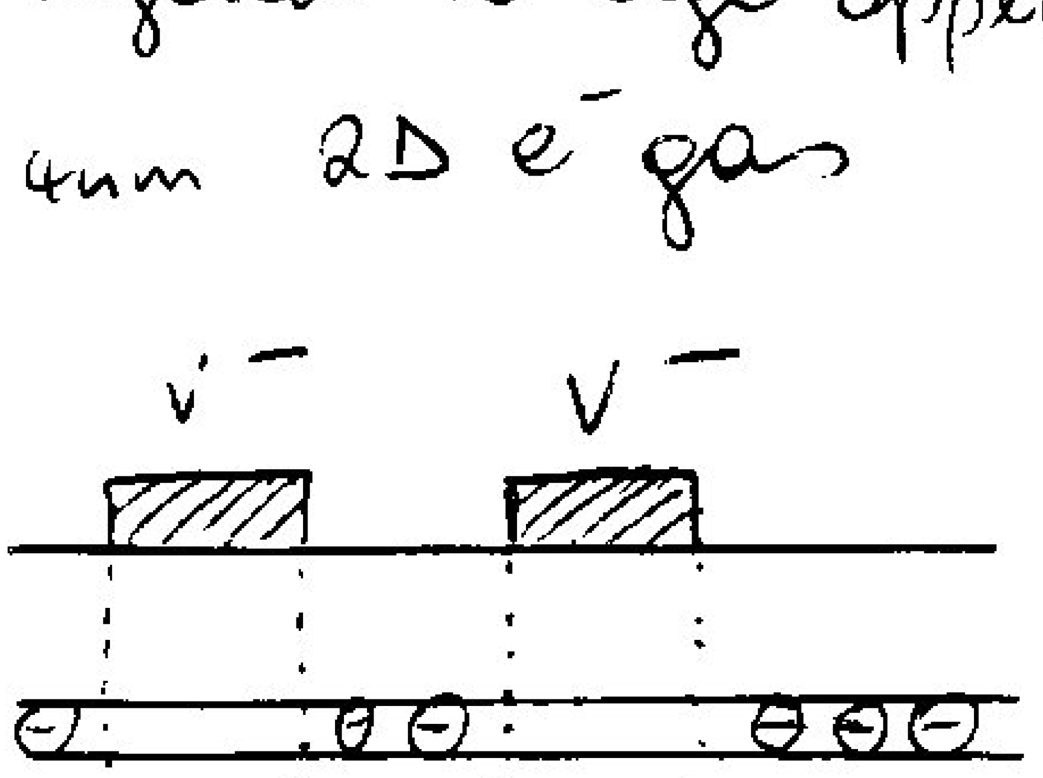
• ELECTRICAL MANIPULATION OF SPIN QUBITS



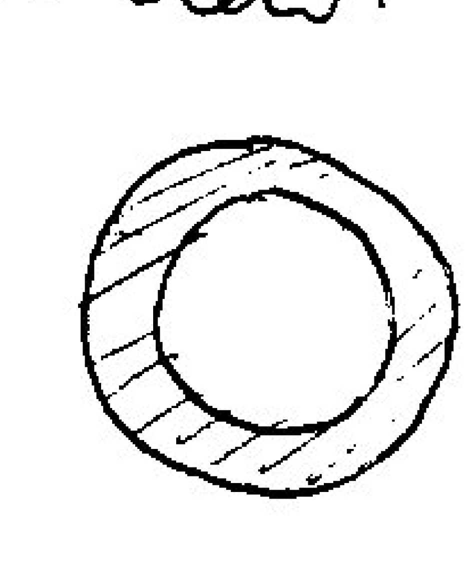
negative voltage applied

4nm 2D e^- gas

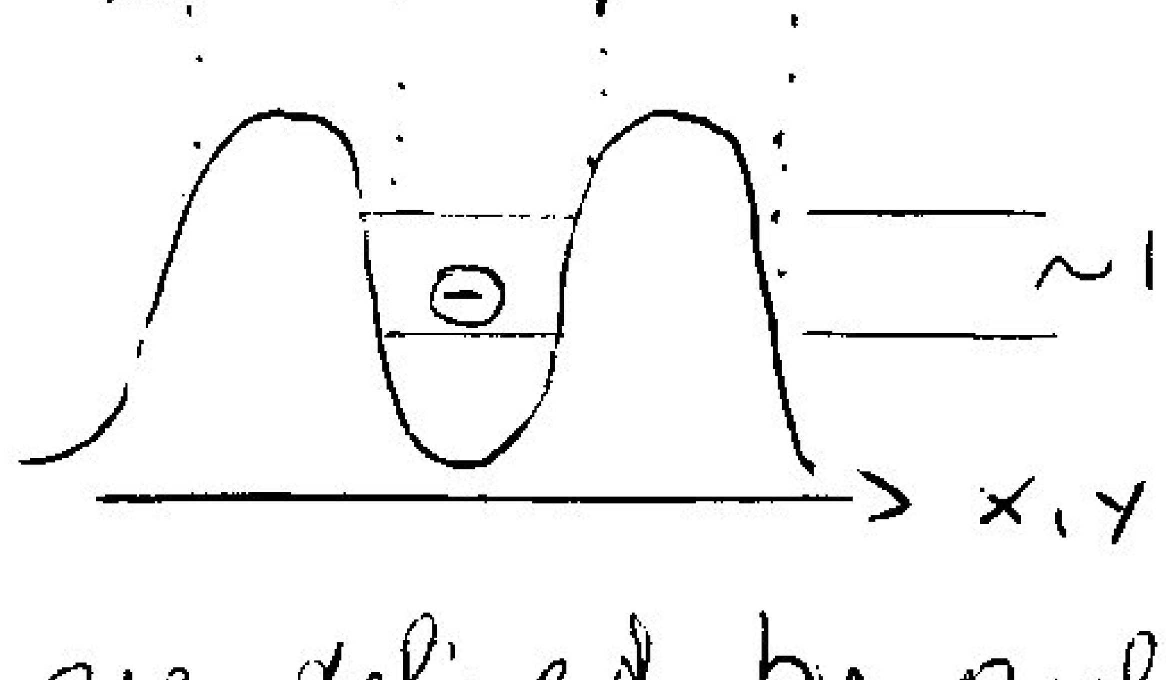
side-view:



top-view:



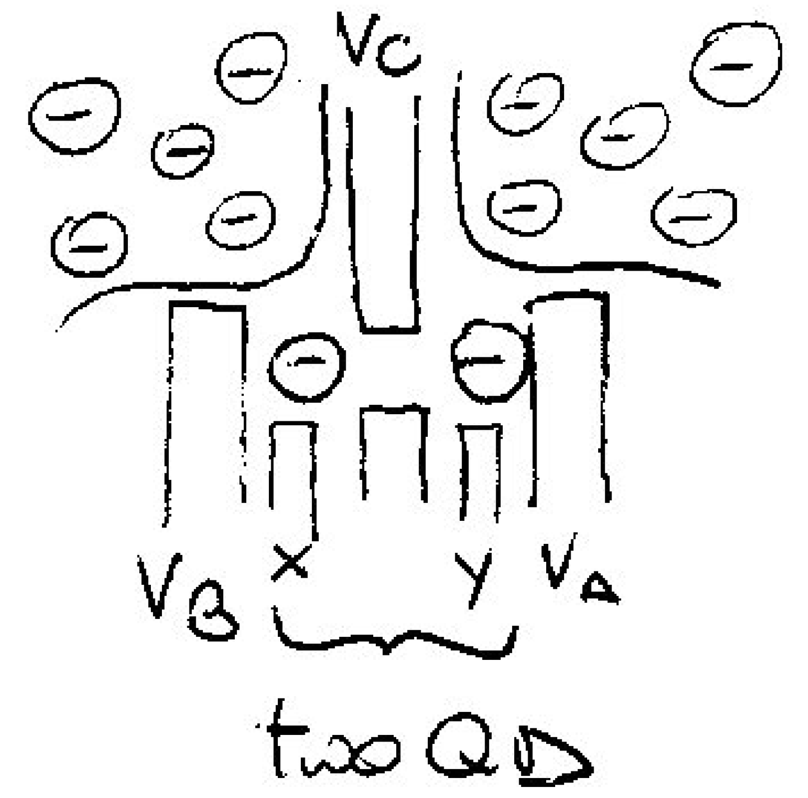
→ potential landscape:

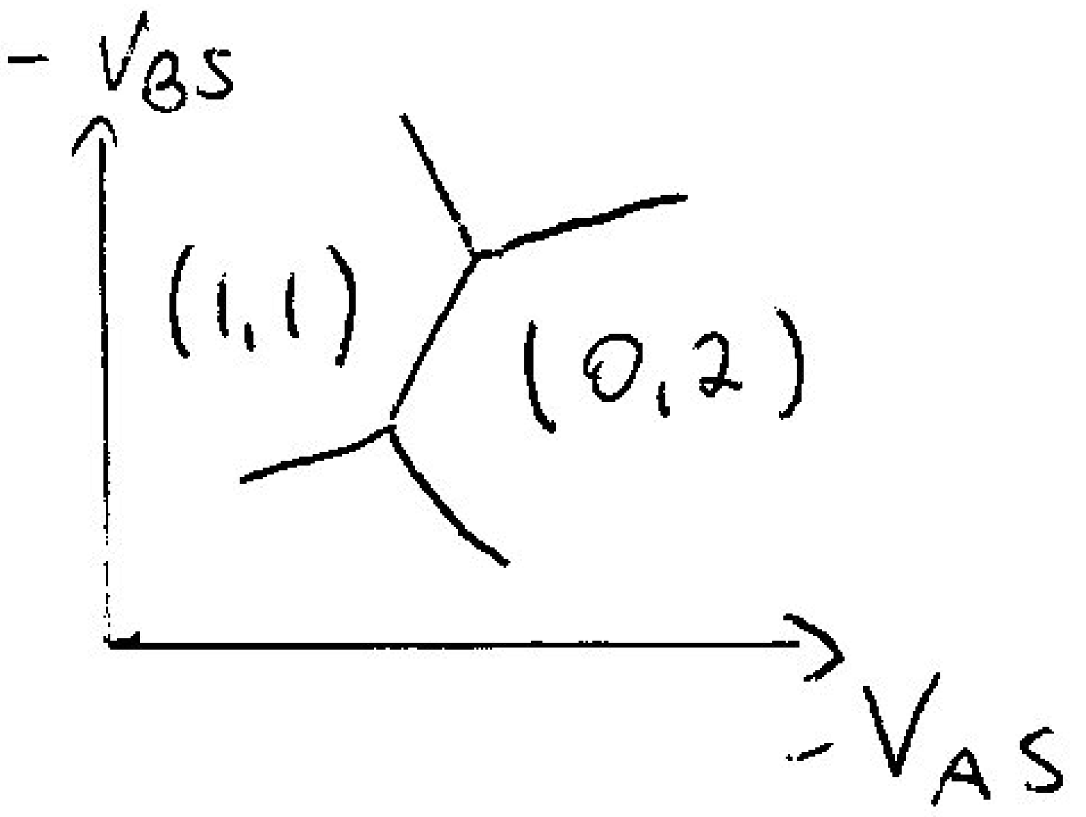


→ thus QD are defined by applied neg. voltage

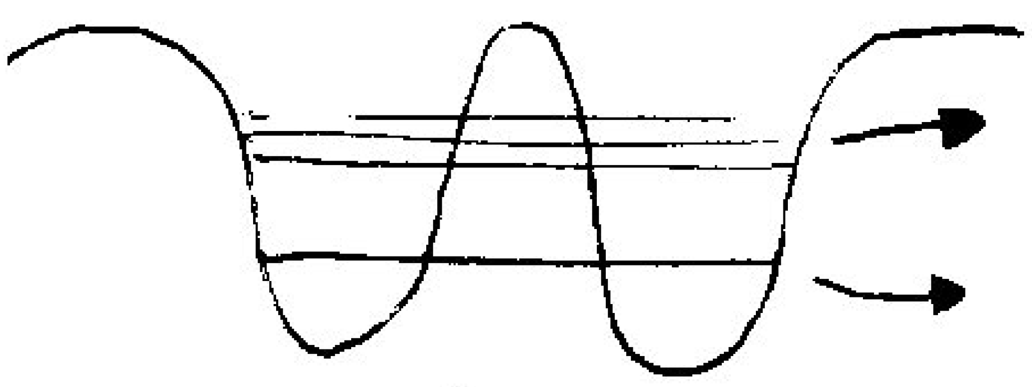
→ <u>Actual structure</u>: (see Science 309, 2005, 2180)

<u>Top view</u>:



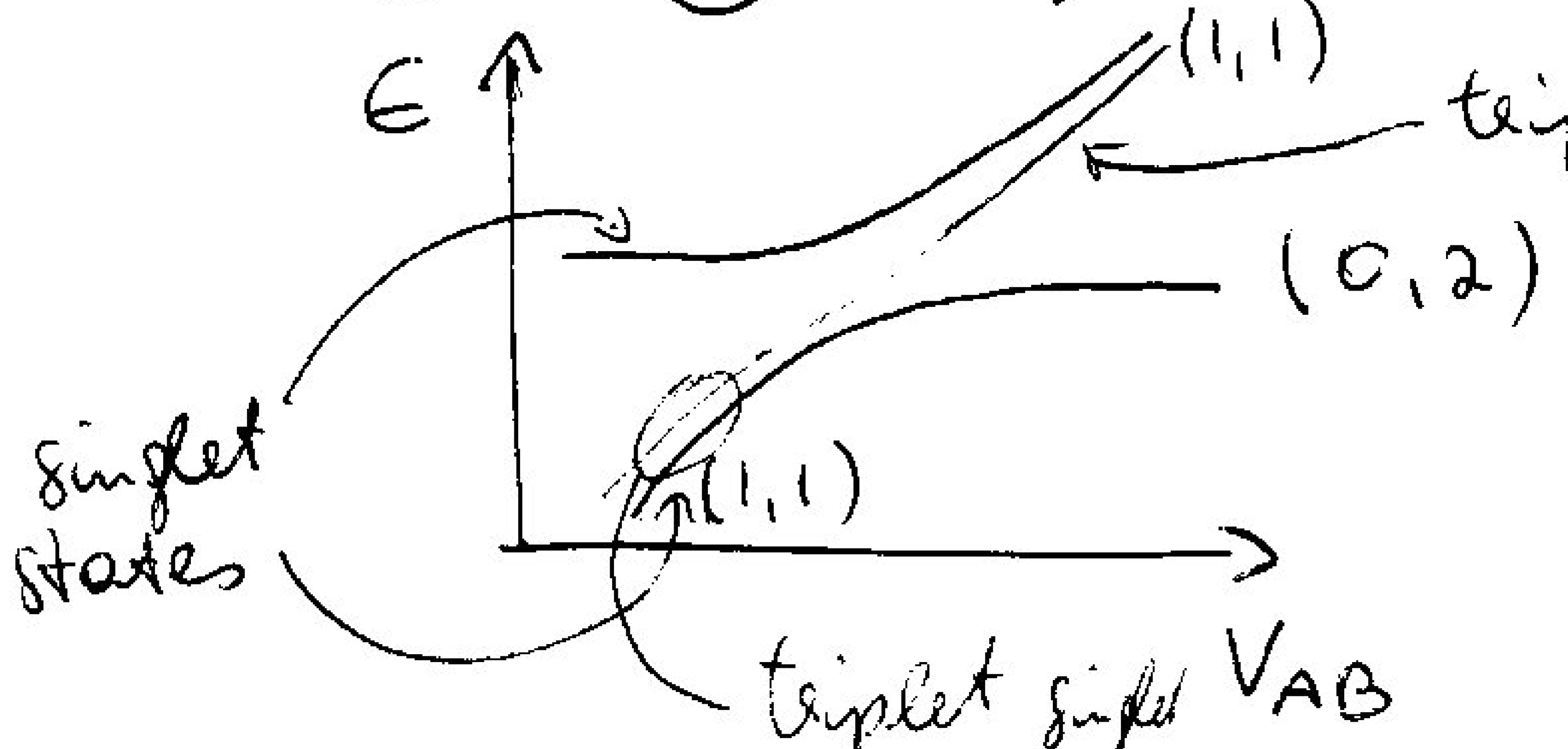


(x, y) no. of e's in x and y position



triplet states $|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle \; \hat{=} \; |1\rangle$

$|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle$ singlet state $\hat{=} \; |0\rangle$



→ splitting can become so small that eigenstates become

$|\uparrow\downarrow\rangle \pm |\downarrow\uparrow\rangle$

→ by applying voltage pulses qubit state can be coherently manipulated: $|\uparrow\downarrow\rangle \leftrightarrow |\downarrow\uparrow\rangle$

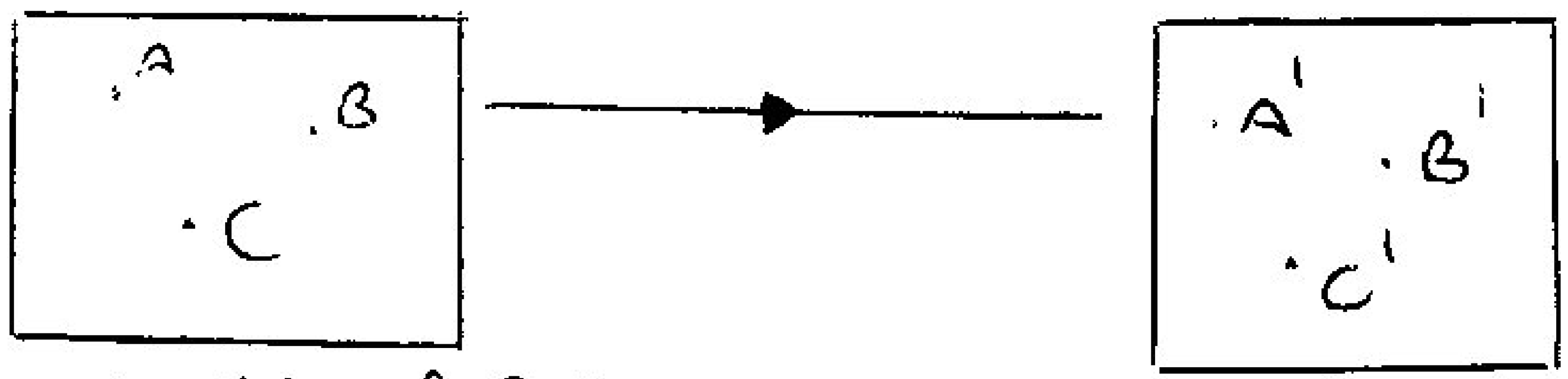
GSIT, Dec. 10th 2007

"Photons + QIP", "cavity QED"

- photons = non-interacting in vacuum
  - ⇒ consequence of Maxwell eq. linearities
  - → ideal for communication + ~~computation~~ (classical + quantum)
  - → awful for computation
- 3 possible applications in quantum information

I) Spin based quantum computer (i.e. 2-level system)

→ photons as means of communication b/w. qubits



node #1 of QC — node #2 of QC

→ achieve quantum communication in distributed QC architecture by means of photonic qubits

spin of qubit A → photon polarization → spin of qubit A'

⇒ cavity QED

$|\psi\rangle_A = \alpha|\downarrow\rangle + \beta|\uparrow\rangle$

$$\boxed{\begin{aligned} &(\alpha|\downarrow\rangle_A + \beta|\uparrow\rangle_A) \otimes |0_p\rangle \otimes |\downarrow\rangle_{A'} \\ \Rightarrow\ & |\downarrow\rangle_A \otimes (\alpha|1_-\rangle + \beta|1_+\rangle) \otimes |\downarrow\rangle_{A'} \\ \Rightarrow\ & |\downarrow\rangle_A \otimes |0\rangle_p \otimes (\alpha|\downarrow\rangle_{A'} + \beta|\uparrow\rangle_{A'}) \end{aligned}}$$

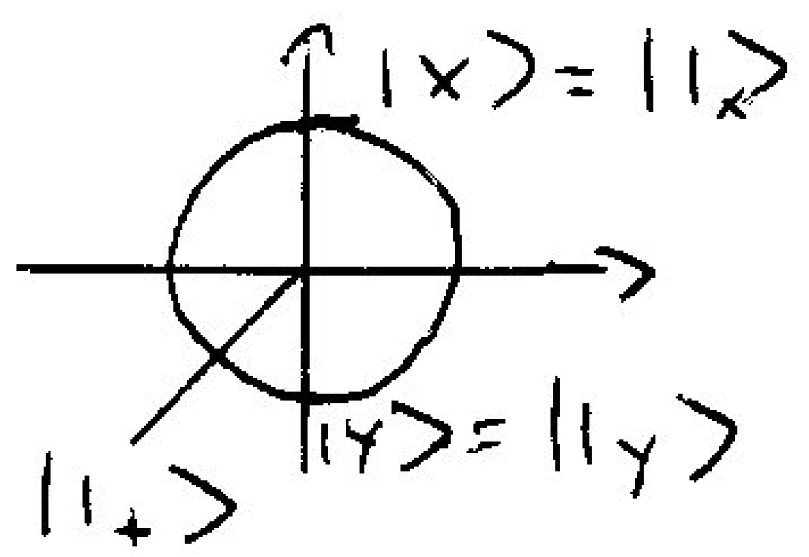
$|n_{+/-}\rangle$ n ... photon # +/- ... polarization

- Types of photonic qubits

1.) Polarization qubits

$|1_-\rangle, |1_+\rangle, |1_x\rangle, |1_y\rangle$, etc.

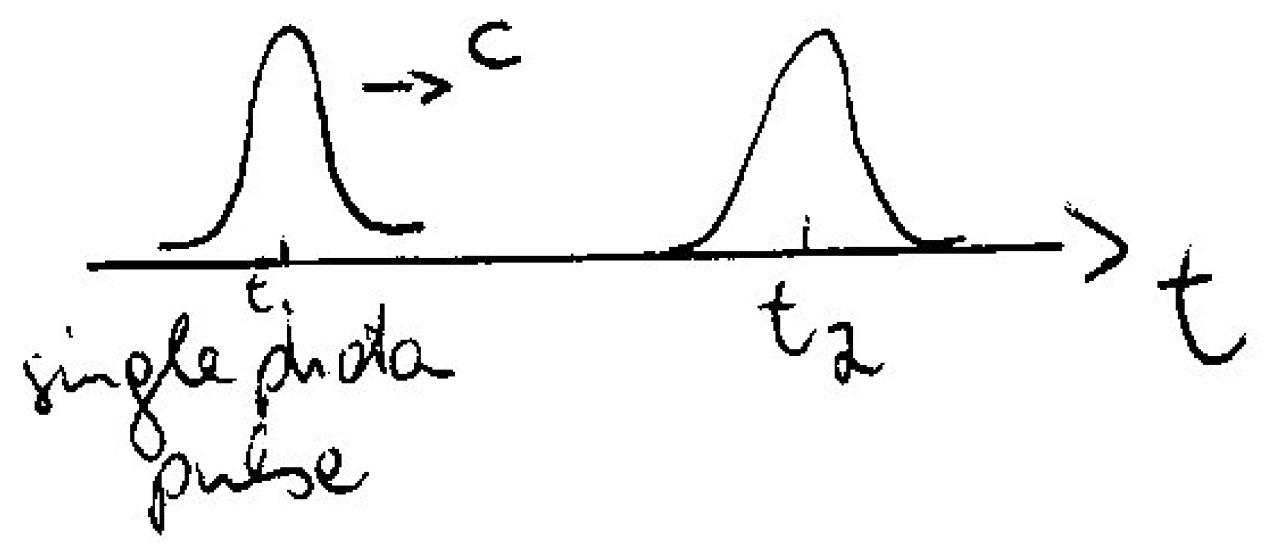
peincare sphere



→ mapping b/w. poincare sphere and Bloch sphere possible

2.) Presence or absence of the photon in a spatial or temporal degree of freedom

a.) Time-bin qubit

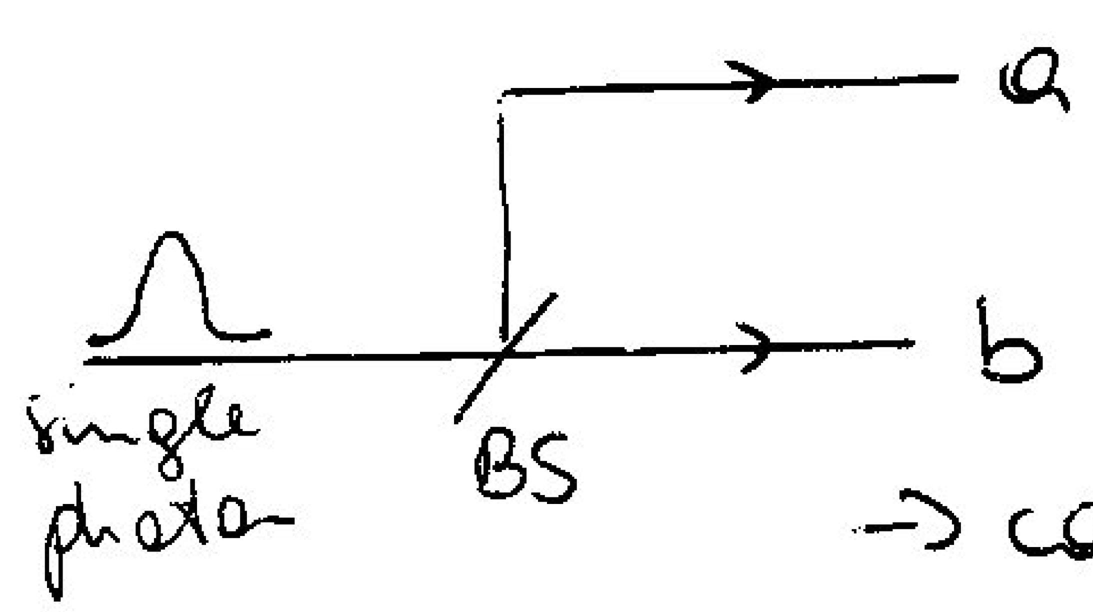


at position z, photon at time t_1 or t_2

$$|\psi\rangle = \alpha\,|1_{t_1}\rangle + \beta\,|1_{t_2}\rangle$$

(infinite dim. Hilbert space)

b.) Dual-rail qubit

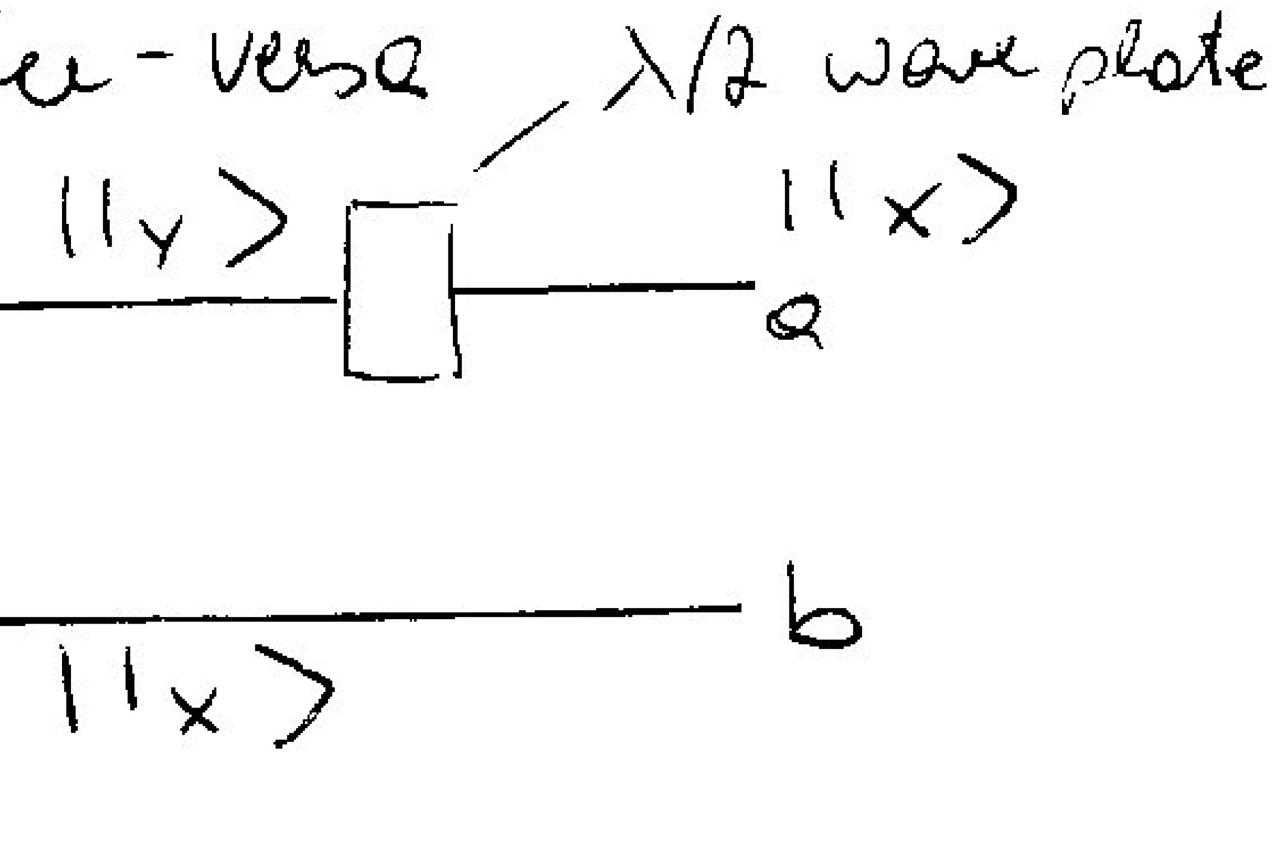


spatial modes a and b

$$|\psi\rangle = \alpha\,|1_a\rangle + \beta\,|1_b\rangle$$

→ can be converted into polarisation qubit and vice-versa

→ Inter conversion:



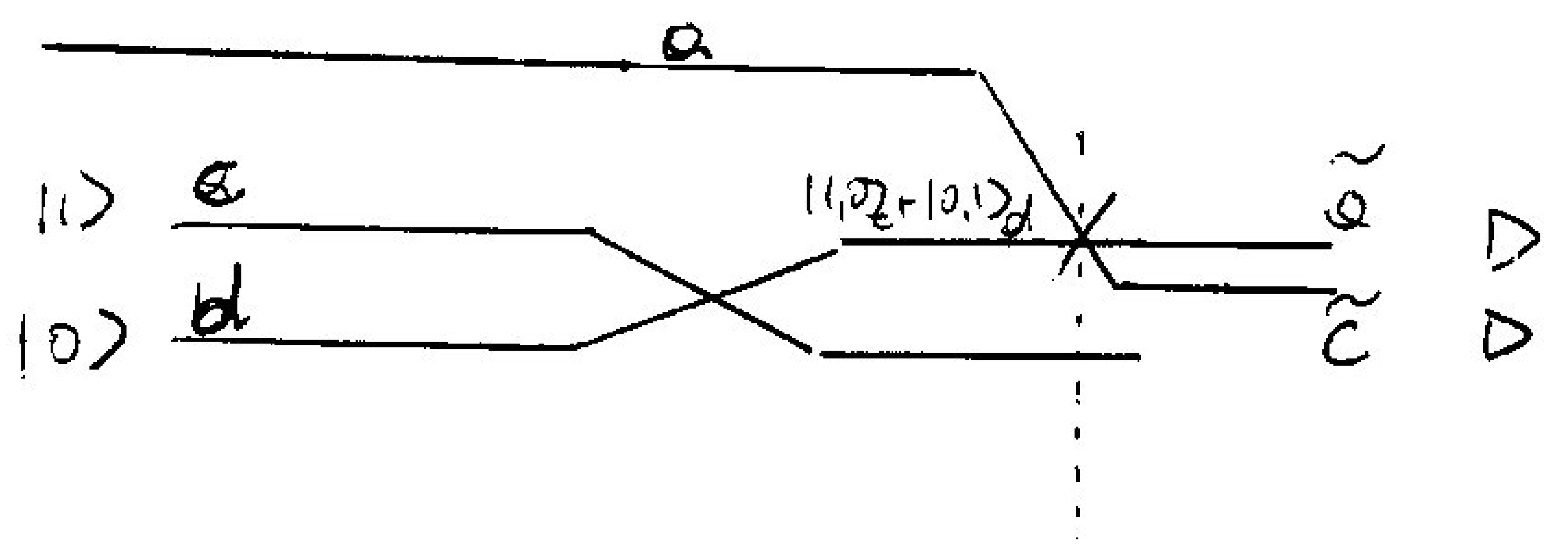
(cavity # states: only useful as quantum memory)

- Advantages a. Disadvantages of photonic qubits

- PROs:
  1. trivial to implement high-efficiency single qubit rotations with linear optics
  2. high effeciency and fast detection (~ ns, much better than for ion traps ~ ms)

- CONs:
  1. difficult to have interactions
  2. quantum memory (or lack of it)

$\alpha |1_a, 0_b\rangle + \beta |0_a, 1_b\rangle$



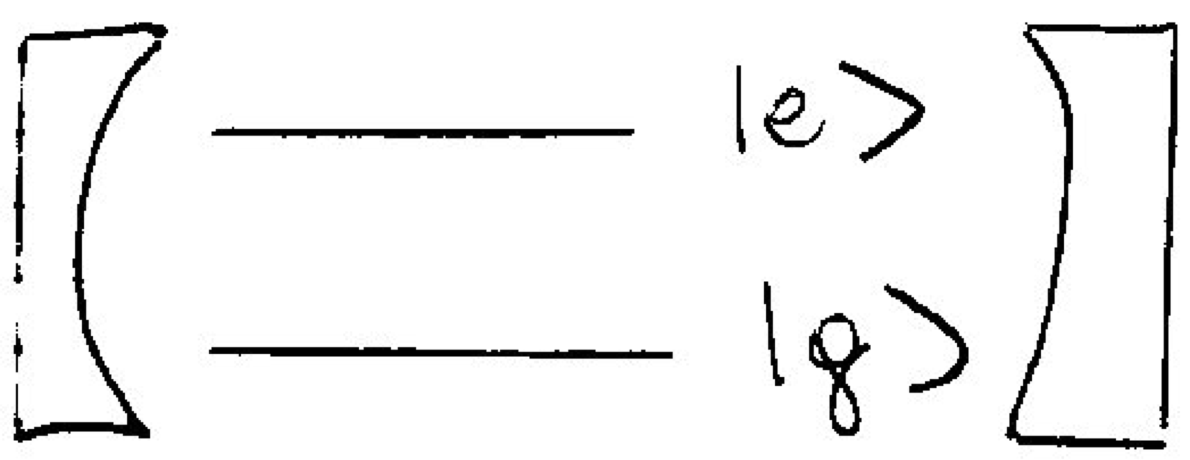
$|1_a, 1_c, 0_d\rangle \rightarrow |2_{\tilde{a}}, 0_{\tilde{c}}, 0_{\tilde{d}}\rangle \pm |0_{\tilde{a}}, 2_{\tilde{c}}, 0_{\tilde{d}}\rangle$

or $|1_a, 0_c, 1_d\rangle \rightarrow |1_{\tilde{a}}, 0_{\tilde{c}}, 1_{\tilde{d}}\rangle \pm |0_{\tilde{a}}, 1_{\tilde{c}}, 1_{\tilde{d}}\rangle$

II.) Photons can act as quantum buses for long-distance coherent interactions → cavity QED

III.) Photonic qubits as qubits in a QC architecture
→ cavity QED: strong interactions between two single photon pulses. $g^2 \gg \kappa, \gamma$

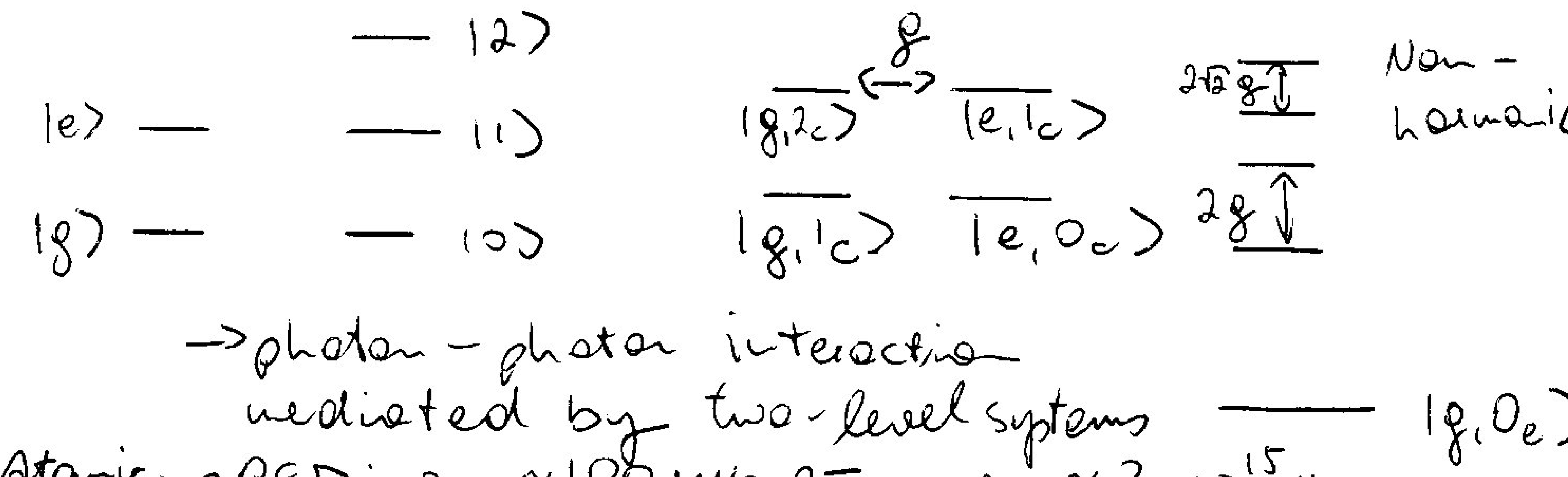
Cavity QED



$$H = \hbar \omega_{eg} \sigma_{ee} + \hbar \omega_c a_c^\dagger a_c + \hbar g_c (\sigma_{eg} a + a^\dagger \sigma_{ge})$$

derivable from first principles
→ electric dipole approx. $\lambda_c \gg d$ ... dipole si

→ RWA



→ photon-photon interaction mediated by two-level systems

- Atomic cQED: $g_e \sim 100$ MHz 2π, $\omega_{eg} \sim 3 \cdot 10^{15}$ Hz
- QD cQED: $g_c \sim 25$ GHz $\times 2\pi$
- circuit QED: $g_c \sim 100$ MHz 2π, $\omega_c \sim 10^{10}$ Hz

- **Atomic & QD**: Fabry - Perot type cavities
min. mode volume

$$V_c = \left(\frac{\lambda}{2n_R}\right)^3$$

$$g_c = \left(\frac{\omega}{2\hbar \varepsilon V_c}\right)^{1/2} \cdot d$$

$$\frac{g_c}{\omega} = \left(\frac{1}{2\hbar\omega \varepsilon V_c}\right)^{1/2} d$$

→ coax - cable : min. mode volume extremely small

→ best possible volume:

$$V_c = d^2 \lambda = d^2 \frac{2\pi c}{\omega_c}$$

$$\rightarrow \frac{g_c}{\omega_c} = \left(\frac{e^2}{4\pi\varepsilon\hbar c}\right)^{1/2} = \sqrt{\alpha}$$

best possible coupling only depends on fundamental constants!

→ other possibilities: plasmonic excitations (limit $\sqrt{\alpha}$ can be beaten)

$$\sqrt{\alpha}\sqrt{\frac{\mu}{\varepsilon}}$$

→ Dicke superradiating phase transition predicted

If we carry out a general meas. but do not look at the outcome:

$p(m)$ for outcome m

$$\rightarrow |\psi'\rangle = \frac{M_m|\psi\rangle}{\sqrt{p(m)}} \qquad \rho_m = \frac{M_m|\psi\rangle\langle\psi|M_m^+}{p(m)}$$

$$\rho = \sum_{m=1}^{N} p(m)\hat{\rho}_m = \sum_m M_m \rho_0 M_m^+ \qquad \rho_0 = |\psi\rangle\langle\psi|$$

state prior to meas.

density operator... max. knowledge possible if outcome unknown.

QSIT, 22.10.2007

Repetition: Generalised quantum measurement

- measurement by means of a set of operators $\{M_m\}$, where m is the measurement outcome
- probability:
$$p(m) = \langle\psi(t)|M_m^+ M_m|\psi(t)\rangle$$
- post-measurement state:
$$|\psi'\rangle = \frac{M_m|\psi(t)\rangle}{\sqrt{p(m)}}$$
- $\sum_m M_m^+ M_m = \mathbb{1}$
- if I fail to note results: $\hat{\rho} = \sum_m M_m \rho_0 M_m^+$

Today: Most-general, not necessarily unitary time evolution

How does reduced density operator of a system A evolve when a larger quantum system $A \otimes B$ evolves unitarily?

larger system: $|\psi_{AB}(t)\rangle$ evolving unitarily with H_{AB}

$$U_{AB}(t) = \exp[-(i/\hbar)H_{AB}t]$$

where $H_{AB} = H_A + H_B + \underbrace{\sum \hat{O}_A \cdot \hat{O}_B}_{H_I} \cdot \alpha$

- if we only focus on system A, we cannot describe it using a state vector:
  - $\rightarrow$ if $\alpha = \emptyset$: $|\psi_{AB}(t)\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$
  - $\rightarrow$ if $\alpha \neq \emptyset$: $|\psi_{AB}(t)\rangle \neq |\psi_A\rangle \otimes |\psi_B\rangle$
- density operator description:
$$\rho_{AB} = |\psi_{AB}\rangle\langle\psi_{AB}|$$

→ define REDUCED density operator of system A:

$$\rho_A = \mathrm{Tr}_B\{\rho_{AB}(t)\}$$
$$= \sum_m \langle m|\rho_{AB}(t)|m\rangle$$

orthonormal basis of system B

- time evolution of ρ_A:

$$|\psi_{AB}(t)\rangle = U_{AB}|\psi_{AB}(0)\rangle$$
$$\hat{\rho}_{AB}(t) = \hat{U}_{AB}\,\hat{\rho}_{AB}(0)\,\hat{U}^+_{AB}$$

- assume $\rho_{AB}(0) = \hat{\rho}_A(0) \otimes \hat{\rho}_B(0)$

$$\overset{!}{=} \hat{\rho}_A(0) \otimes |0\rangle_B\,{}_B\langle 0|$$

i.e. interaction switched off until $t=0$.

$$\rightarrow \hat{\rho}_{AB}(t) = U_{AB}\left(\hat{\rho}_A(0)\otimes|0\rangle_B\,{}_B\langle 0|\right)U^+_{AB}$$

$$\rightarrow \boxed{\hat{\rho}_A(t)} = \sum_m \underbrace{\langle m|\hat{U}_{AB}|0\rangle_B}_{\hat{M}_m \text{ operator acting on system A}}\,\hat{\rho}_A(0)\,{}_B\langle 0|U^+_{AB}|m\rangle$$

$$\boxed{= \sum_m \hat{M}_m\,\hat{\rho}_A(0)\,\hat{M}^+_m}$$

same form as in last lecture!

$$\sum_m M_m M_m^+ = \sum_m \langle m|\hat{U}_{AB}|0\rangle_B\,{}_B\langle 0|U^+_{AB}|m\rangle$$

$$= \cancel{\sum_m |0\rangle_B\,{}_B\langle 0|\,\langle m|\underbrace{U_{AB}U^+_{AB}}_{1\!\!1}|m\rangle}$$

$$= \underbrace{\sum_m |m\rangle\langle m|}_{1\!\!1}\;\underbrace{{}_B\langle 0|U^+_{AB}U_{AB}|0\rangle_B}_{=1}$$

$$= 1\!\!1$$

→ Thus: Decoherence is the same as a measurement where we don't know the results. Decoherence is described by the interaction ~~of~~ with System B.

Evolution in $\hat{\rho}_A(t)$

1) Maps a Hamiltonian matrix to another
2) preserves trace (ρ_A should have unity trace)
3) it preserves positivity

→ describes a proper density operator evolution

- Kraus - Operators:

$$N_n = \underbrace{U_{nm}}_{\substack{\text{unitary} \\ \text{basis-set transf.}}} M_m \qquad \{N_n, M_m\} \text{ kraus operators}$$

The basis used for the measurement operator is only unique if there is no interaction between A and B.

- QUANTUM OPERATIONS (see also 29.10.)

= most general in which an open quantum system evolves

→ Map from an initial density operator to a final one
$(t = \emptyset)$ $(t = t')$

$$\hat{\rho}_A(t) = \Lambda(\hat{\rho}_A(0)) \qquad \Lambda \text{ ... superoperator, generalization of } M_m$$

1.) Trace should be preserved $Tr(\rho_A(0)) = 1$
→ $Tr(\rho_A(t)) = 1$

$$\rho_A(t) = \sum_i p_i |\psi_i(t)\rangle\langle\psi_i(t)| \qquad (I)$$

2.) Λ should map Hermitian operators onto Hermitian operators

3.) Λ has to be linear, since QM is linear

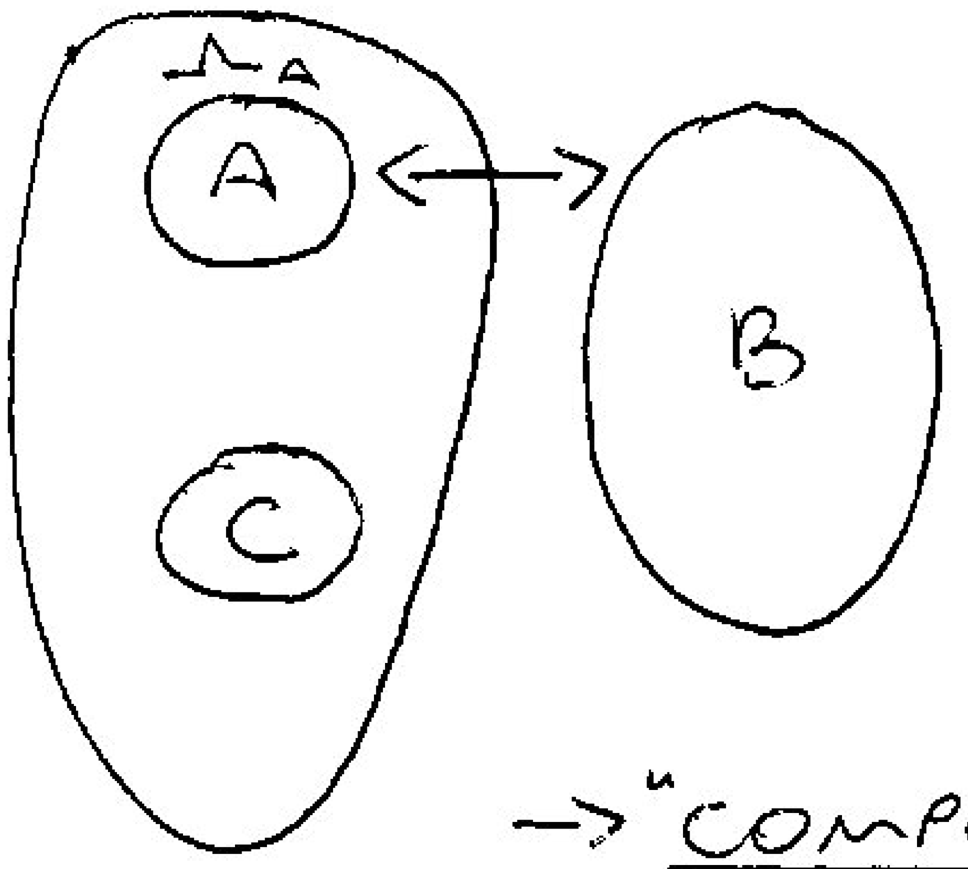
$$\Lambda(\lambda\rho_1 + (1-\lambda)\rho_2) = \lambda_1\Lambda(\rho_1) + (1-\lambda_1)\Lambda(\rho_2)$$

4.) Since according to (I), the eigenvalues of ρ_A can in principle be negative, but since neg. probs are not allowed: Λ should be completely positive, i.e. if $\rho_A(0)$ is positive, the $\Lambda(\rho_A(0))$ is also positive.

a.) $\rho_A(0) \otimes \rho_C(0)$ → condition 4.) sufficient

b.) entanglement bw. A and C

$\Lambda_A \otimes \mathbb{1}_C$ be positive for all C's



$$\Lambda_A \otimes \mathbb{1}_C = \Lambda_{AC}$$

$(\Lambda_A \otimes \mathbb{1}_C)(\rho_{AC}(0)) = \rho_{AC}(t)$ has to be positive

→ "COMPLETE POSITIVITY" required

<u>example: Transpose Operation</u>

$$\rho = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \rho^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

if there were no entanglement, then transposing would still give positive EVs; but if there is entanglement, transposition of one system would lead to negative EVs. Thus: <u>Transpose Operation is unphysical</u>

→ <u>Kraus's representation theorem:</u>
Any physical evolution that satisfies conditions (1)-(4) can be represented as:

$$\boxed{\rho_A(t) = \Lambda(\rho_A(0)) = \sum_m M_m \rho_A(0) M_m^\dagger}$$

as discussed before.

- <u>General evolution of a Qubit</u> ($\equiv$ Decoherence)

ρ_A = 2×2 matrix so that M_m also 2×2 matrices in computational basis $\{|0\rangle, |1\rangle\}$.

<u>Note:</u> any 2×2 matrix can be written as a superposition of the Pauli-Matrices σ_i, $i = x, y, z$ + identity

• <u>Decoherence process #1: Bit-Flip channel</u>

with prob. p, qubit flips $0 \to 1$, $1 \to 0$ and with prob. $(1-p)$ nothing happens.

$$\rho_A(0) \to \boxed{BF} \to \rho_A(t_F)$$

$$\rho_A(t_F) = M_0 \rho_A(0) M_0^\dagger + M_1 \rho_A(0) M_1^\dagger$$

$M_0 = \sqrt{p}\,\hat{X}$ $\qquad M_1 = \sqrt{1-p}\,\mathbb{1}$

• **DP #2: Phase-Flip**

with prob. p relative phase flips $|0\rangle + |1\rangle \to |0\rangle - |1\rangle$

$M_0 = \sqrt{p}\,\hat{Z}$ $\qquad M_1 = \sqrt{1-p}\,\mathbb{1}$

Note: M_i has infinitely many representations:
e.g. if $p = 1/2$ then $U_{mn} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ so that

$$\underbrace{N_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{\text{proj. onto } |0\rangle} \qquad \underbrace{N_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{\text{projection onto } |1\rangle} \qquad \text{Kraus operator}$$

just like a measurement

22.10.07 – QSIT EXERCISE

wingeum@phys.ethz.ch

- wavefunction collapse
- projective (von Neumann) measurement:
  An observable is given by a hermitian operator $A = A^\dagger$. The EVs λ_i are the possible outcomes of the measurement with the post-measurement state $|i\rangle$.
  → LINK to generalised measurement

- Note: since $A = A^\dagger$, $\{|i\rangle\}$ is a CONS and thus spectral decomps.

$$\hat{A} = \sum_i |i\rangle\langle i|\,\alpha_i = \sum_i \lambda_i \hat{P}_i$$

-> measurement by means of P_i
-> post-measurement state

$$|\psi'\rangle = \frac{P_i|\psi\rangle}{\sqrt{\langle\psi|P_i|\psi\rangle}}$$

-> generalised measurement: $P_i \to M_i$

- Photon # Measurement

mode $(\vec{k}_m \alpha_n)$ $\qquad \mathcal{H} = \{|n\rangle\}$ Hilbert space

$\hat{N}|n\rangle = a^\dagger a|n\rangle = n|n\rangle$

-> Measurement operator: $P_n = |n\rangle\langle n|$

- <u>Problem</u>: ∄ ideal, non-destructive photon counter
(e.g. Photo-Multiplier)
→ projective description not suited

→ $M_n = |0\rangle\langle n|$ photon state destroyed so that $|0\rangle$

→ generalised measurement postulate

- <u>POVMs</u>:
Since we're not interested in the post-measurement state here, only the product in $p(n) = \langle\psi|\underbrace{M_n^\dagger M_n}_{\hat{E}_n}|\psi\rangle$ is required!

$$\sum_n \hat{E}_n = \mathbb{1} \quad \text{with all } E_n \text{ positive.}$$

- <u>Example</u>: Distinguishing quantum states
  <u>i.)</u> $|\psi_1\rangle \perp |\psi_2\rangle$ → projective measurement possible
  → states can be distinguished with perfect reliability
  <u>ii.)</u> non-orthogonal states: $\langle\psi_2|\psi_1\rangle \neq \emptyset$
  → distinguishing can't be done with perfect reliability

→ Assume Alice gives Bob qubits either in state $|0\rangle$ or ~~?~~

or $|\psi_2\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

$|\psi_1\rangle = |0\rangle$

→ Bob can do POVM measurement:

$$\hat{E}_1 = \frac{\sqrt{2}}{1+\sqrt{2}}|1\rangle\langle 1|$$

$$\hat{E}_2 = \frac{\sqrt{2}}{1+\sqrt{2}}\frac{(|0\rangle - |1\rangle)(\langle 0| - \langle 1|)}{2}$$

$$\hat{E}_3 = \mathbb{1} - \hat{E}_1 - \hat{E}_2$$

<u>a.)</u> initial state $|\psi_1\rangle = |0\rangle$

$\langle\psi_1|\hat{E}_1|\psi_1\rangle = \emptyset$ → state was $|\psi_2\rangle$

$\langle\psi_1|\hat{E}_2|\psi_1\rangle = 1 - \frac{1}{\sqrt{2}} \approx 30\%$