Quantum Systems for Information Technology

winter semester 06/07

Structure

- introduction to quantum information processing
  - quantum mechanics reminder
  - qubits, qubit dynamics
  - entanglement, teleportation
  - Bell inequalities
  - measurement, decoherence
  - quantum algorithms (Deutsch-Jozsa, factoring, searching)

- quantum systems for information technology
  - ions and neutral atoms in electromagnetic traps
  - nuclear magnetic resonance in molecules and solids
  - charges and spins in semiconductor quantum dots
  - charges and flux quanta in superconducting electronic circuits

- selected topics of current research

Guest Lectures

- ion trap quantum computing, Hartmut Haeffner, Innsbruck
- quantum dots
- nuclear spins

Time and Place

- lecture: Monday, 14:45 – 16:30, HCI D 2
- exercises: Tuesday, 13:45 – 14:30, HCI H 8

- are there timing conflicts with other lectures?

Credit Requirements

- attend lectures

- attend problem solving classes
- solve problem sets and present solutions to problems in class at least twice

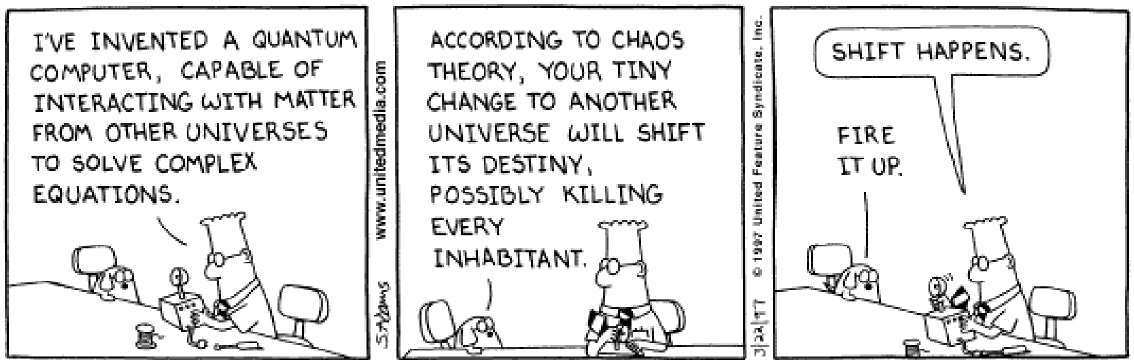
- do a 25 minute presentation on a topic of current research in Quantum Information Science to the class.
- topics will be chosen from a selection of research papers to be made available later during the semester.

Exams

- Sessionspruefung (summer 2007)

- other dates possible for mobility students
  (contact your mobility program advisor)

Fire it up!


I'VE INVENTED A QUANTUM COMPUTER, CAPABLE OF INTERACTING WITH MATTER FROM OTHER UNIVERSES TO SOLVE COMPLEX EQUATIONS.
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S.Adams
ACCORDING TO CHAOS THEORY, YOUR TINY CHANGE TO ANOTHER UNIVERSE WILL SHIFT ITS DESTINY, POSSIBLY KILLING EVERY INHABITANT.
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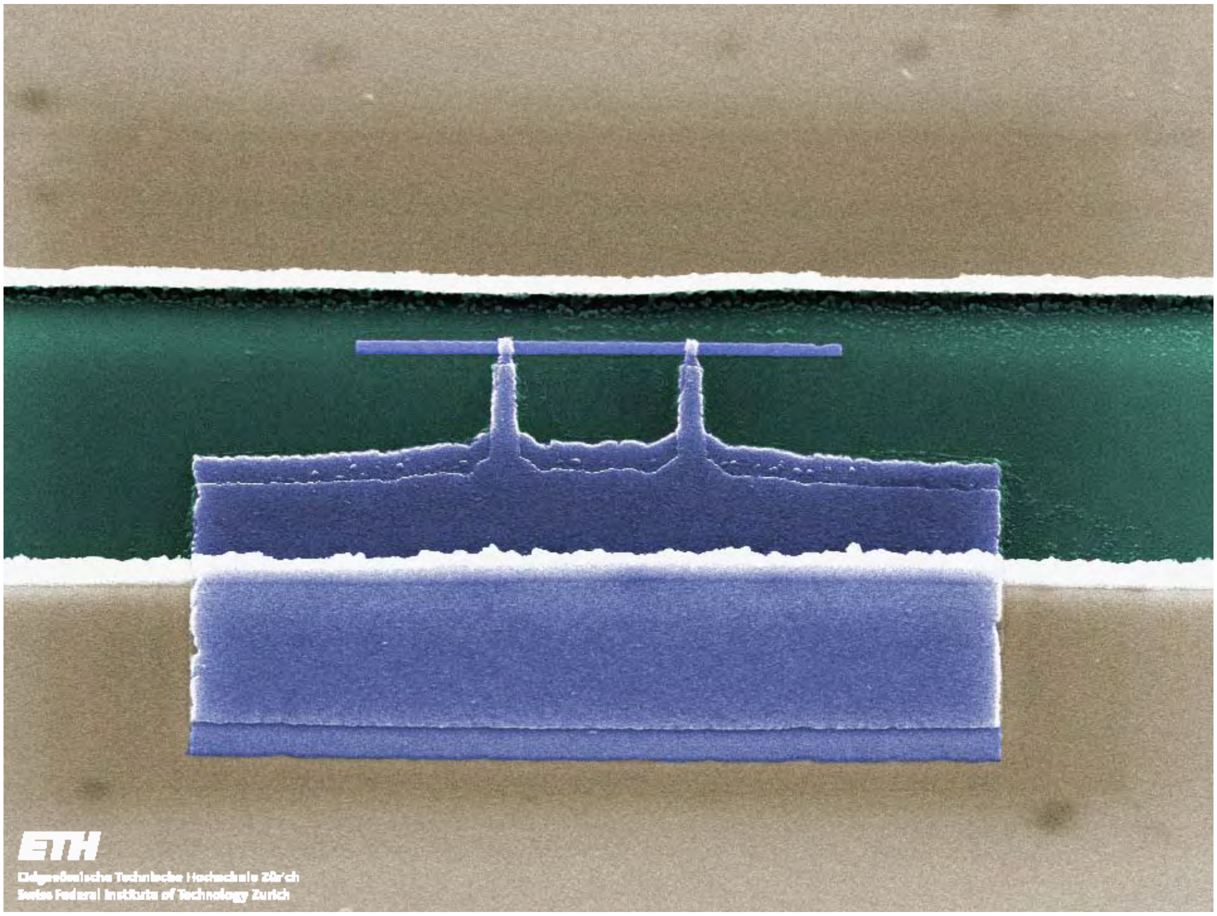
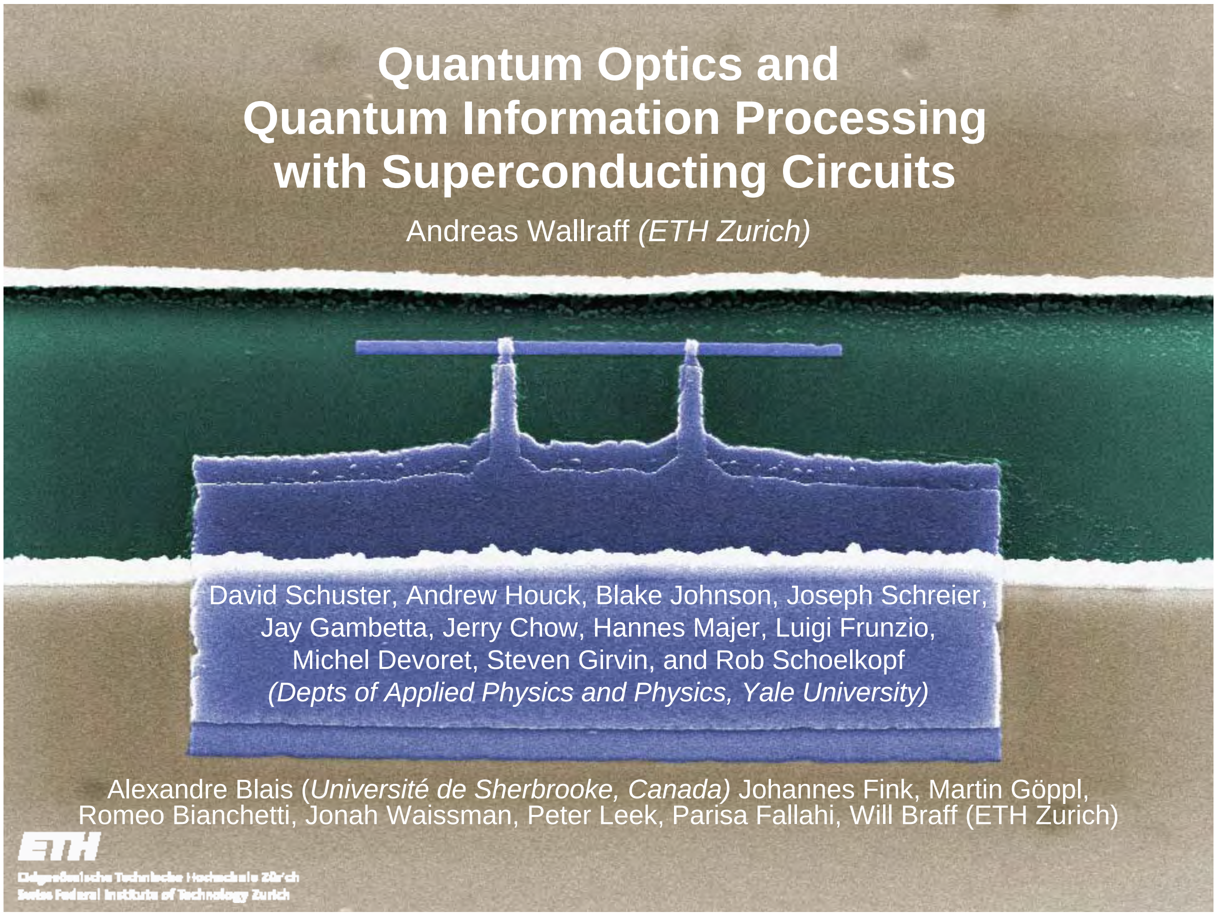
3/22/97
SHIFT HAPPENS.
FIRE IT UP.



Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich



Quantum Optics and Quantum Information Processing with Superconducting Circuits

Andreas Wallraff *(ETH Zurich)*

David Schuster, Andrew Houck, Blake Johnson, Joseph Schreier,
Jay Gambetta, Jerry Chow, Hannes Majer, Luigi Frunzio,
Michel Devoret, Steven Girvin, and Rob Schoelkopf
(Depts of Applied Physics and Physics, Yale University)

Alexandre Blais (*Université de Sherbrooke, Canada)* Johannes Fink, Martin Göppl,
Romeo Bianchetti, Jonah Waissman, Peter Leek, Parisa Fallahi, Will Braff (ETH Zurich)

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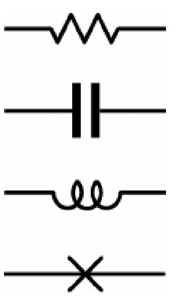
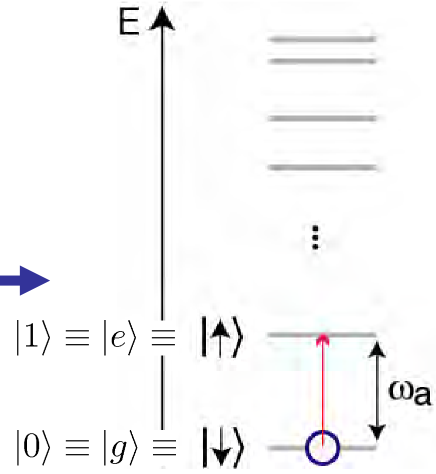
Swiss Federal Institute of Technology Zurich

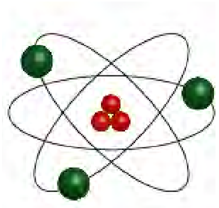
Quantum Electrical Circuits (Qubits)

circuit elements:



macroscopic artificial atom:





two level system
$\equiv$ qubit

$\sim 10\,\mathrm{GHz} \sim 0.5\,\mathrm{K}$

ingredients:

- nonlinearity
- low temperatures
- small dissipation
- isolation from environment

use as basic element for solid state quantum information processor

Review: M. H. Devoret, A. Wallraff and J. M. Martinis, *condmat/0411172* (2004)

Non-Classical Features of Quantum Systems

- **superpositions (one qubit)**

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

- **quantum parallelism (n qubits)**

$$|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \otimes ... \otimes |\psi_n\rangle$$

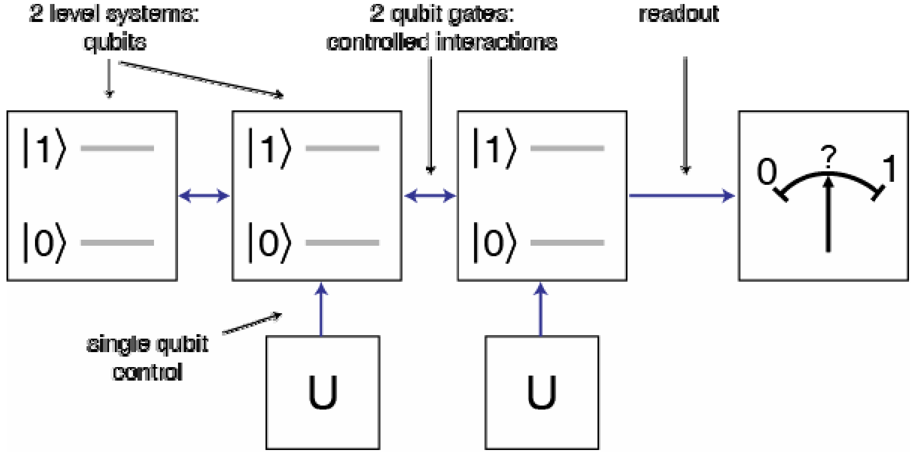
- **entanglement (non-classical correlations)**

$$|\psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

make use of these features for novel approaches to information processing

Quantum Information Processing

schematic of a generic quantum processor:



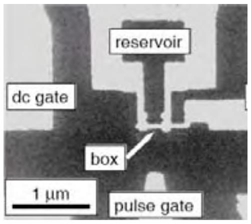
long term goal:
- process information (decrypt, search, ...)

current goals:
- control quantum systems and their interactions
- understand measurement process and decoherence

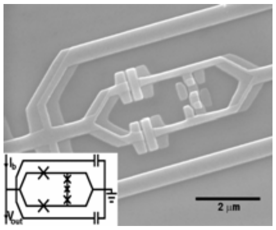
M. Nielsen, I. Chuang, Quantum Computation and Quantum Information (Cambridge, 2000).

Realizations of Quantum Electrical Circuits

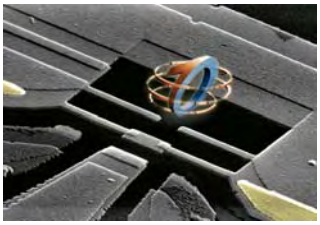
'artificial atoms'– single superconducting qubits



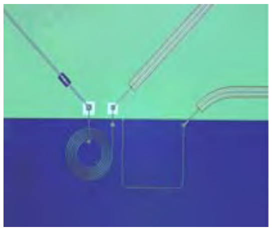
NEC, Japan
Nature **398**
(1999)



TU Delft, Netherlands
Science **285, 290, 299**
(1999, 2000, 2003)

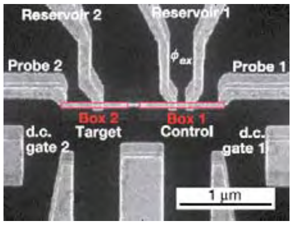


CEA Saclay, France
Science **296** (2002)

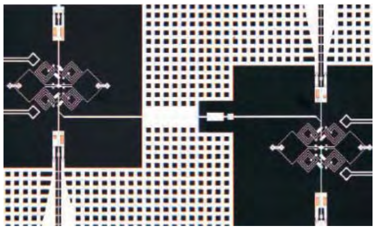


NIST, UCSB
Phys. Rev. Lett. **89, 93**
(2002, 2004)

'artificial molecules'– coupled superconducting qubits



NEC, Japan
Nature **421**, **425**
(2003, 2003)

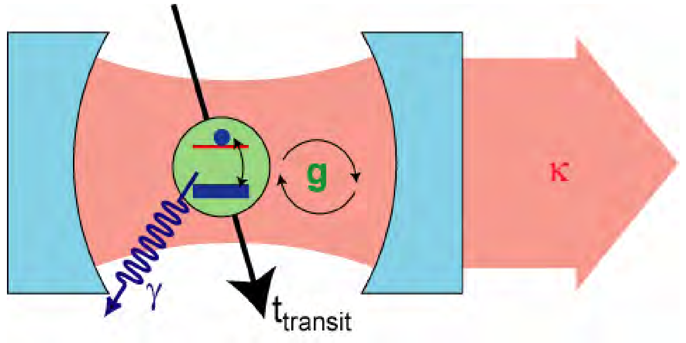


NIST, UCSB
Science **307**
(2005)

recent review:
G. Wendin and V.S. Shumeiko
cond-mat/0508729 (2005)

atomic or molecular physics with circuits

Cavity Quantum Electrodynamics (CQED)



Jaynes-Cummings Hamiltonian

$$H = \hbar\omega_r \left(a^\dagger a + \frac{1}{2} \right) + \frac{\hbar\omega_a}{2}\sigma^z + \hbar g(a^\dagger\sigma^- + a\sigma^+) + H_\kappa + H_\gamma$$

strong coupling limit $(g = dE_0/\hbar > \gamma,\ \kappa\,,1/t_{\mathrm{transit}})$

D. Walls, G. Milburn, Quantum Optics (Spinger-Verlag, Berlin, 1994)

Motivation

make realm of quantum optics accessible in superconducting circuits

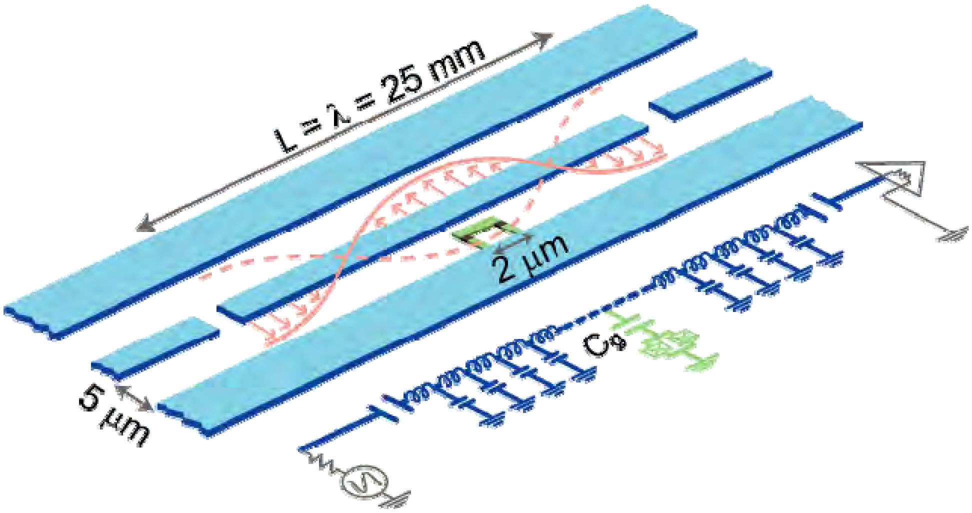
- exploit strong matter/light interactions in solids
- inspired by atomic cavity QED

circuit quantum electrodynamics (QED)

- new solid state architecture for quantum information processing
  → controlled qubit photon interaction
  → quantum interfaces / hybrid systems

A. Blais, R.-S. Huang, A. Wallraff, S. M. Girvin, and R. J. Schoelkopf,
PRA **69,** 062320 (2004)

Circuit QED Architecture

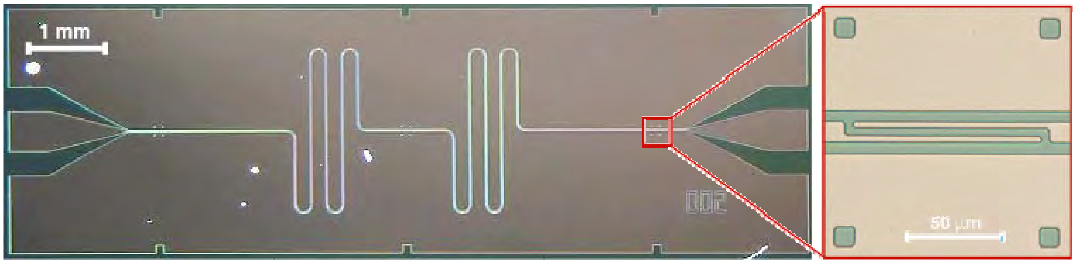


elements

- the cavity: a superconducting 1D transmission line resonator (large E_0)
- the artificial atom: a Cooper pair box (large d)

A. Blais, R.-S. Huang, A. Wallraff, S. M. Girvin, and R. J. Schoelkopf,
PRA **69,** 062320 (2004)

1D Transmission Line Cavity



harmonic oscillator

$$H_r = \hbar\omega_r\left(a^\dagger a + \frac{1}{2}\right)$$

large vacuum field

$$V_{0,\mathrm{rms}} = \sqrt{\frac{\hbar\omega_r}{2C}} \approx 1\,\mu\mathrm{V}$$

$$E_0 = \frac{V_{0,\mathrm{rms}}}{b} \approx 0.2\,\mathrm{V/m}$$

$\times 100$ larger than E_0 in 3D microwave cavity

control photon lifetime

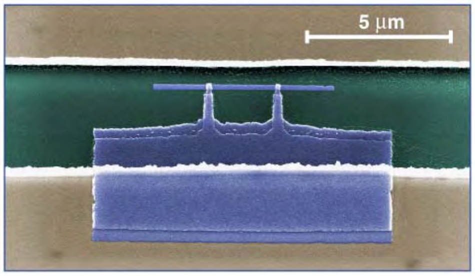
photon decay rate:

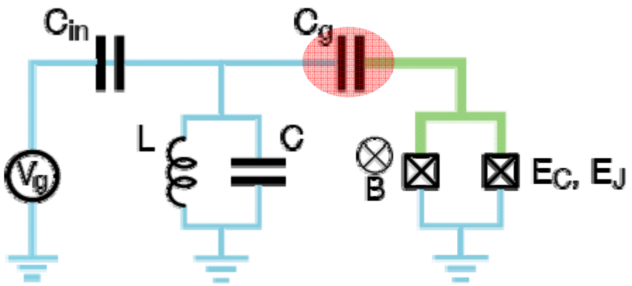
$$\frac{\kappa}{2\pi} = \frac{\nu_r}{Q} \approx 0.8\,\mathrm{MHz}$$

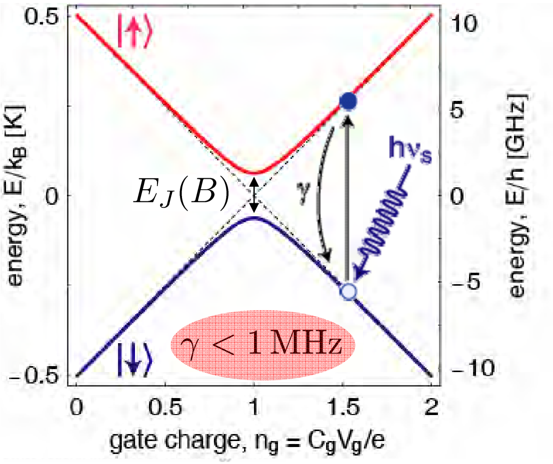
photon lifetime:

$200\,\mathrm{ns}$

The Cooper Pair Box







qubit Hamiltonian (2-state approx.):

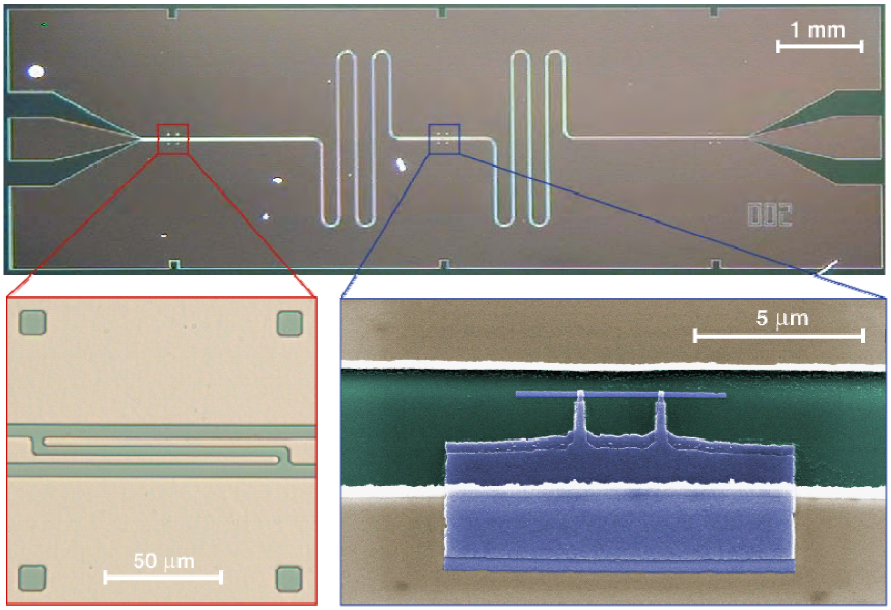
coupling strength: $\sigma^z)$

$$\hbar g = eV_{0,\mathrm{rms}}\frac{C_g}{C_\Sigma}$$

$$\Longrightarrow \nu_{\mathrm{vac}} = \frac{g}{\pi} \approx 1\ldots100\,\mathrm{MHz}$$

$$E_J = E_{J,\max}\cos\left(\pi\Phi_b\right)$$

Realization



superconducting cavity QED circuit

A. Wallraff, D. Schuster, ..., S. Girvin, and R. J. Schoelkopf, *Nature (London)* **431**, 162 (2004)

ETH
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Swiss Federal Institute of Technology Zurich

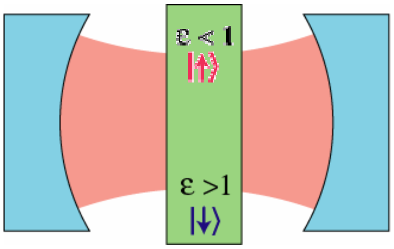
Qubit Measurement: Non-Resonant Interaction

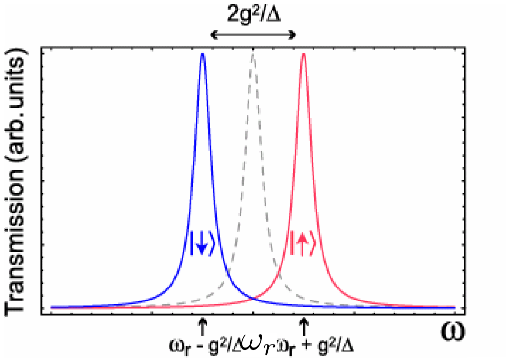
approximate diagonalization for $|\Delta| = |\omega_a - \omega_r| \gg g$

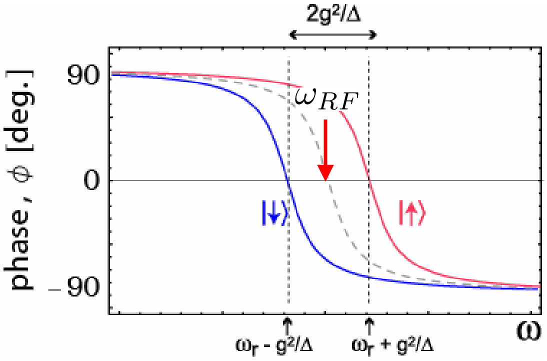
$$H \approx \hbar\left(\omega_r + \frac{g^2}{\Delta}\sigma_z\right)a^\dagger a + \frac{1}{2}\hbar\left(\omega_a + \frac{g^2}{\Delta}\right)\sigma_z$$

//
cavity frequency shift
***and* qubit ac-Stark shift**

//
Lamb shift

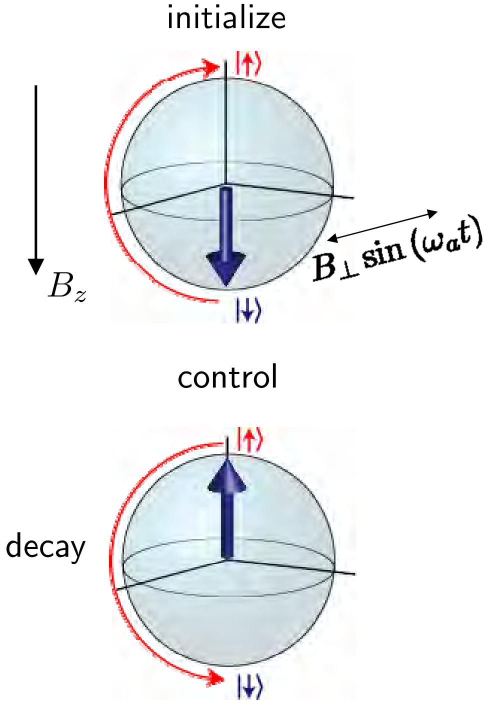


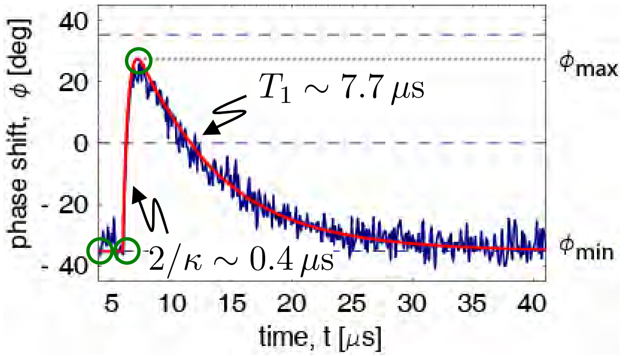




A. Blais *et al.*, *PRA* **69,** 062320 (2004)

Time-Resolved Dispersive QND Readout



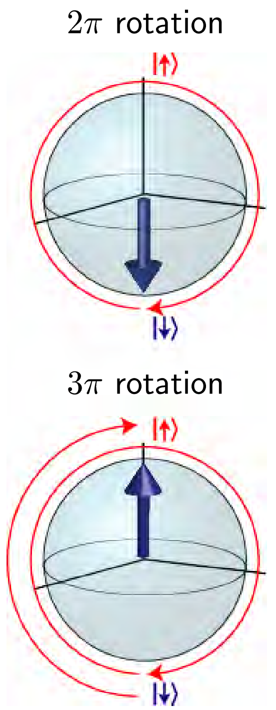


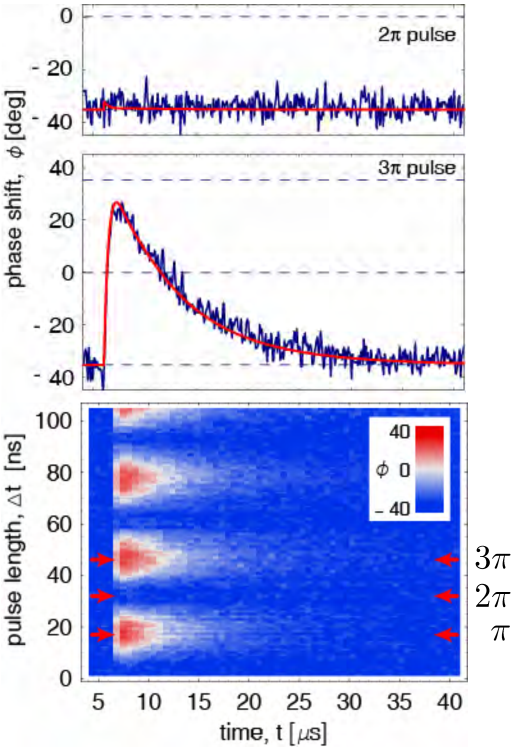
measurement properties:

- **continuous**
- **dispersive**
- **quantum non-demolition**
- **in good agreement with predictions**

Wallraff, Schuster, Blais, ... Girvin, and Schoelkopf,
Phys. Rev. Lett. **95**, 060501 (2005)

Varying the Control Pulse Length



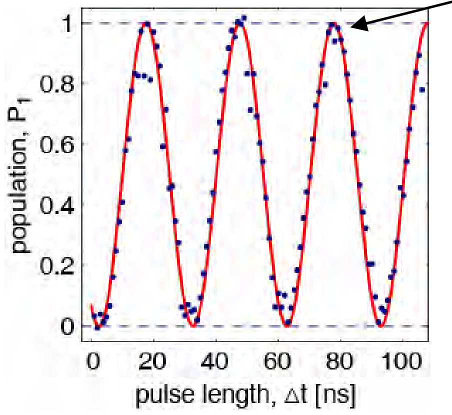


Wallraff, Schuster, Blais, ... Girvin, and Schoelkopf, *Phys. Rev. Lett.* **95**, 060501 (2005)

High Visibility Rabi Oscillations

Rabi oscillations:

visibility $95 \pm 5\%$



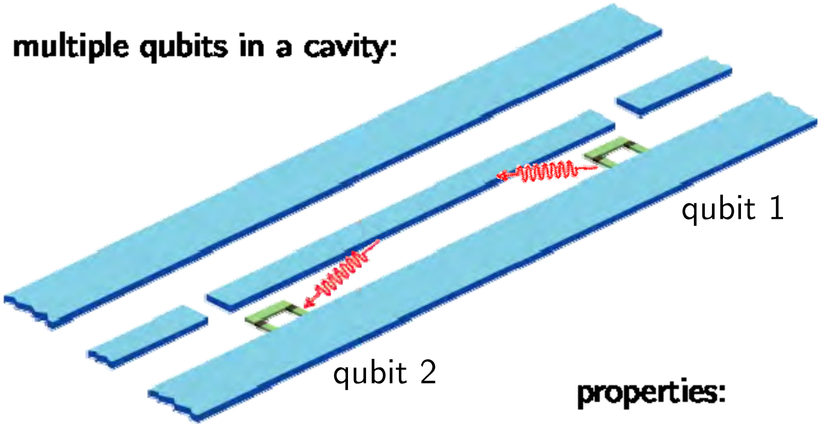
for superconducting qubits:

- first high visibility
- well characterized and understood measurement
- good control accuracy

A. Wallraff, D. I. Schuster, A. Blais, L. Frunzio, J. Majer,
S. M. Girvin, and R. J. Schoelkopf, *Phys. Rev. Lett.* **95**, 060501 (2005)

Coupled Qubits

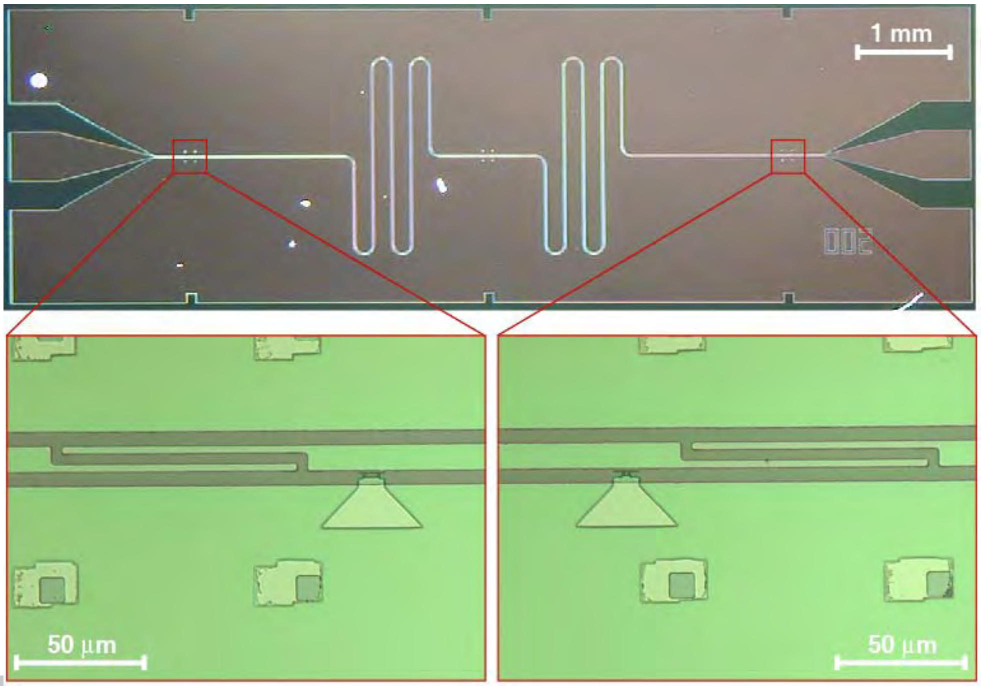
multiple qubits in a cavity:



properties:

- well characterized interaction
- controllable
- non-local
- scalable

Coupled Qubit Device

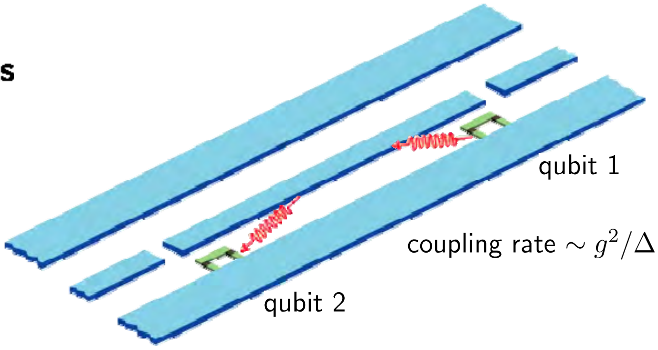


Planned Operation of Two Qubit Gate (CNOT)

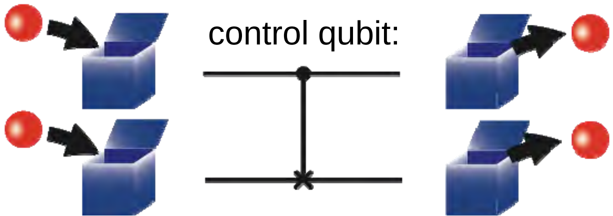
coupling through photons

$|0\rangle \equiv |g\rangle \equiv$ 🔴

$|1\rangle \equiv |e\rangle \equiv$ 🟢



qubit-qubit interaction (CNOT, XOR):


control qubit:

target qubit:

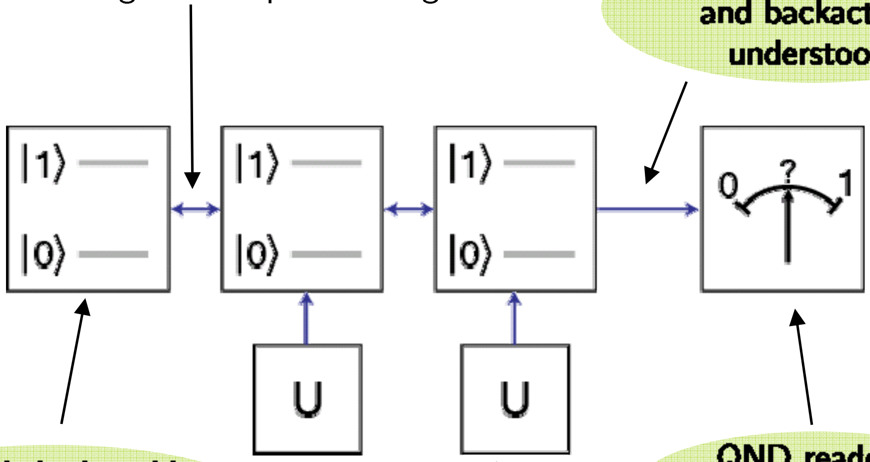
truth table

IN	OUT
🔴 🔴	🔴 🔴
🔴 🟢	🔴 🟢
🟢 🔴	🟢 🟢
🟢 🟢	🟢 🔴

Circuit QED and Quantum Computation

next steps:
use cavity as non-local inter-qubit coupling bus
demonstrate 2 bit gate and quantum algorithm

measurement process and backaction understood



realized single qubits with long coherence times

demonstrated accurate control

QND readout with high visibility

New ETH Zurich Quantum Device Team


Johannes Fink


Romeo Bianchetti


Martin Göppl


Peter Leek


Parisa Fallahi


Jonah Waissman


Will Braff


Gaby Strahm


H.-R. Aeschbach


Andreas Wallraff

with funding from:

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Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

FNSNF
Schweizerischer Nationalfonds zur Förderung
der wissenschaftlichen Forschung
Fonds national suisse de la recherche scientifique
Swiss National Science Foundation
Fondo nazionale svizzero per la ricerca scientifica

new lab started in April 2006

Quantum Systems for Information Technology

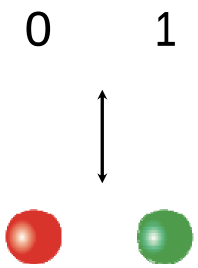
a brief introduction

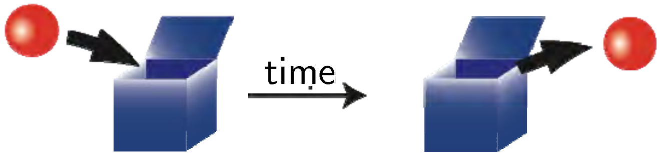
A Brief Introduction

- classical information
- quantum information
  - logic gates
  - entanglement
  - ‘no cloning theorem’
  - teleportation
  - quantum parallelism
- experimental realizations

Classical Information

basic unit of classical information: the bit





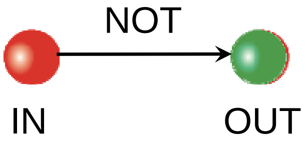
properties of a classical bit:

- easy to write
- easy to read out
- easy to copy
- extremely stable

Classical Logic Gates

single bit gate:

bit:

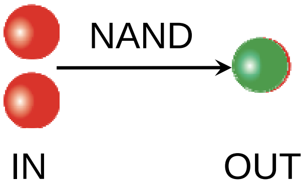


IN	OUT
🔴	🟢
🟢	🔴

universal 2-bit gate:

bit 1:

bit 2:



IN	OUT
🔴 🔴	🟢
🔴 🟢	🟢
🟢 🔴	🟢
🟢 🟢	🔴

irreversible

Quantum Information

fundamental unit of quantum information

qubit
(qu**an**tum **bit**) {0,1} ⟶ {🔴, 🟢}

superposition principle:

a 0 + b 1 ⟶ a 🔴 + b 🟢

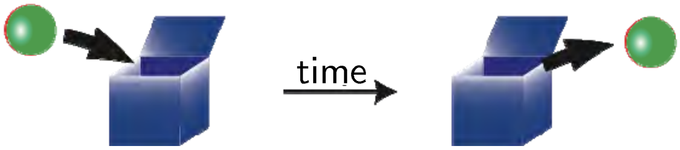
probability amplitudes

$P(0) = |a|^2$, $P(1) = |b|^2$

$|a|^2+|b|^2 = 1$

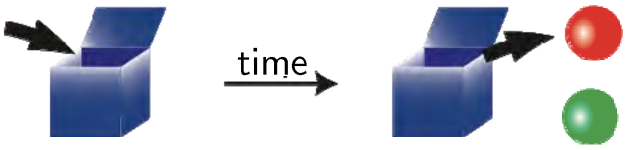
Measurement of a Qubit

in basis state:



in superposition state:

a 🔴 + b 🟢



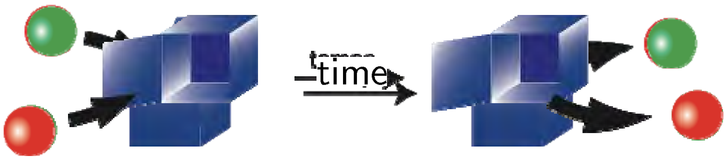
P(🔴) = $|a|^2$

P(🟢) = $|b|^2$

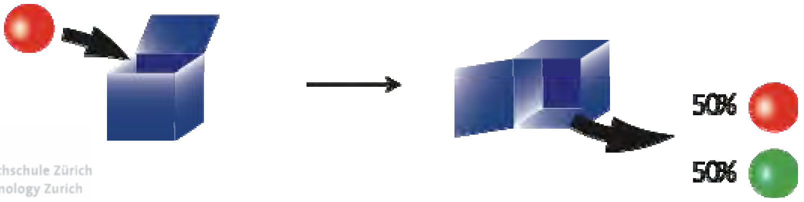
Measurement of different properties

e.g. polarization: { ↕ , ↔ } { ⤢ , ⤡ }

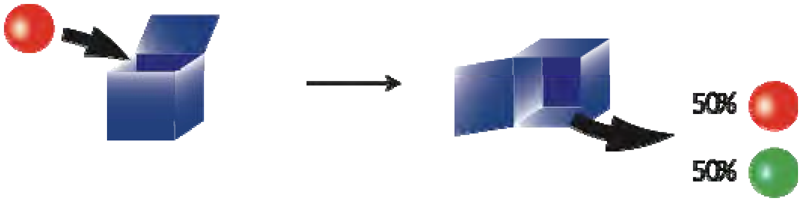
preparation and measurement of the same property :



preparation of one and measurement of a different property :



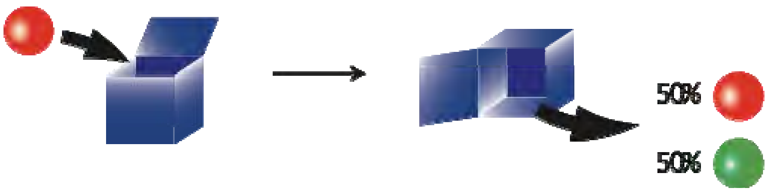
Some strange properties of quantum objects



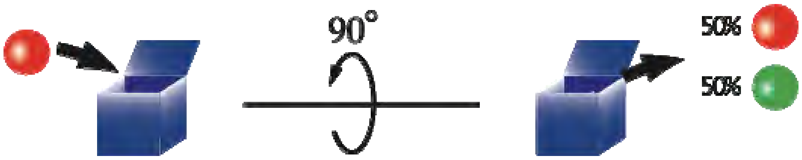
Probabilistic result:	it is fundamentally impossible to predict outcome of all measurements with certainty
state of quantum object before measurement	superposition of states 🔴 **and** 🟢
state of quantum object after measurement:	classical result 🔴 **or** 🟢

Quantum Logic Operation on a Single Qubit

Generation of superposition states:

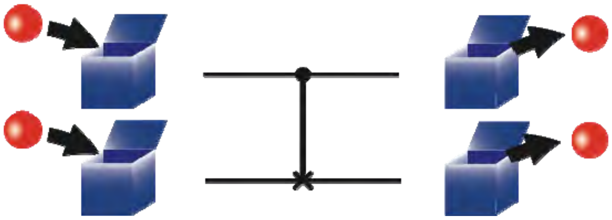


Rotate qubit by 90 degrees:



Quantum Logic Operation on Two Qubits: CNOT

qubit-qubit interaction (CNOT, XOR):



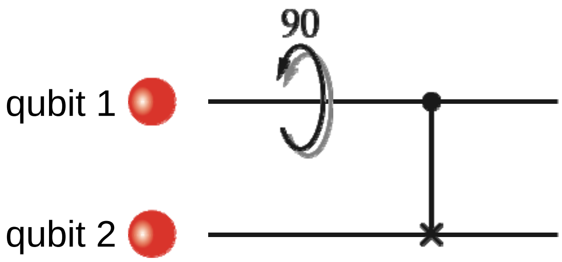
control qubit:

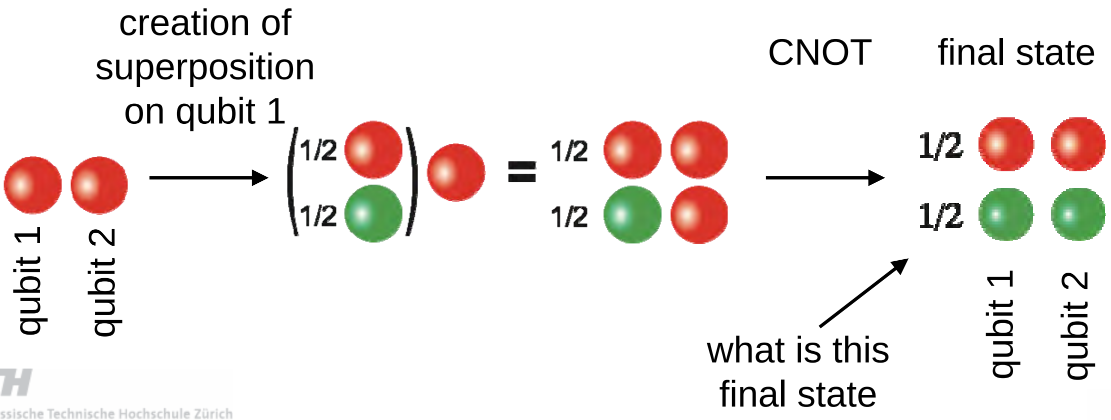
target qubit:

CNOT truth table:

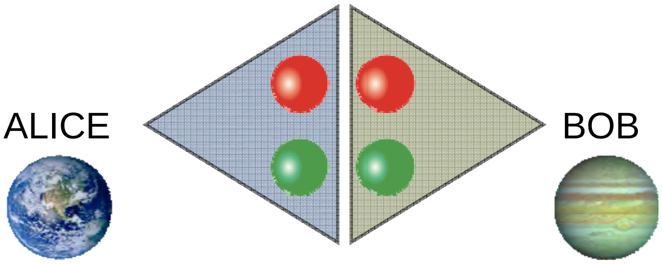
IN	OUT
🔴 🔴	🔴 🔴
🔴 🟢	🔴 🟢
🟢 🔴	🟢 🟢
🟢 🟢	🟢 🔴

Quantum Logic Circuit

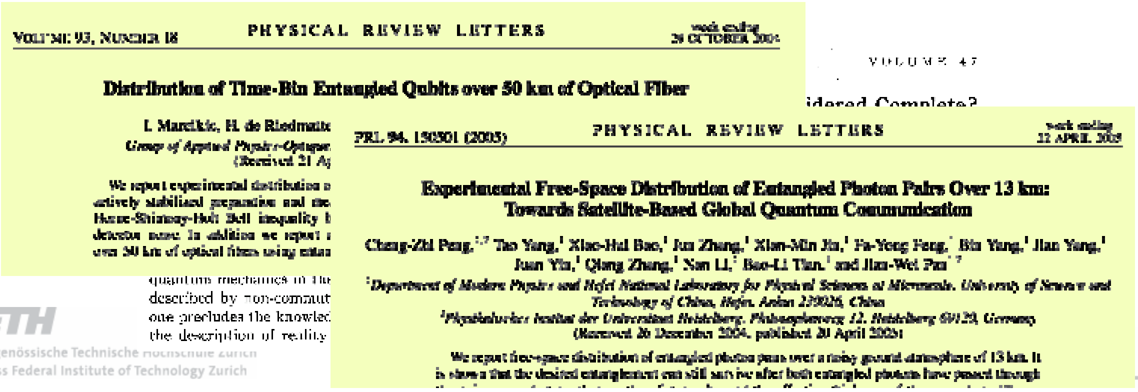




Entanglement



Alice’s measurement on her qubit
determines the state of Bobs qubit ... instantaneously


VOLUME 93, NUMBER 18 PHYSICAL REVIEW LETTERS week ending 29 OCTOBER 2004

Distribution of Time-Bin Entangled Qubits over 50 km of Optical Fiber

I. Marcikic, H. de Riedmatte

VOLUME 47

idered Complete?

PRL 94, 150501 (2005) PHYSICAL REVIEW LETTERS week ending 22 APRIL 2005

**Experimental Free-Space Distribution of Entangled Photon Pairs Over 13 km:
Towards Satellite-Based Global Quantum Communication**

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(Received 26 December 2004; published 20 April 2005)

We report free-space distribution of entangled photon pairs over a noisy ground atmosphere of 13 km. It is shown that the desired entanglement can still survive after both entangled photons have passed through

Features of Entanglement

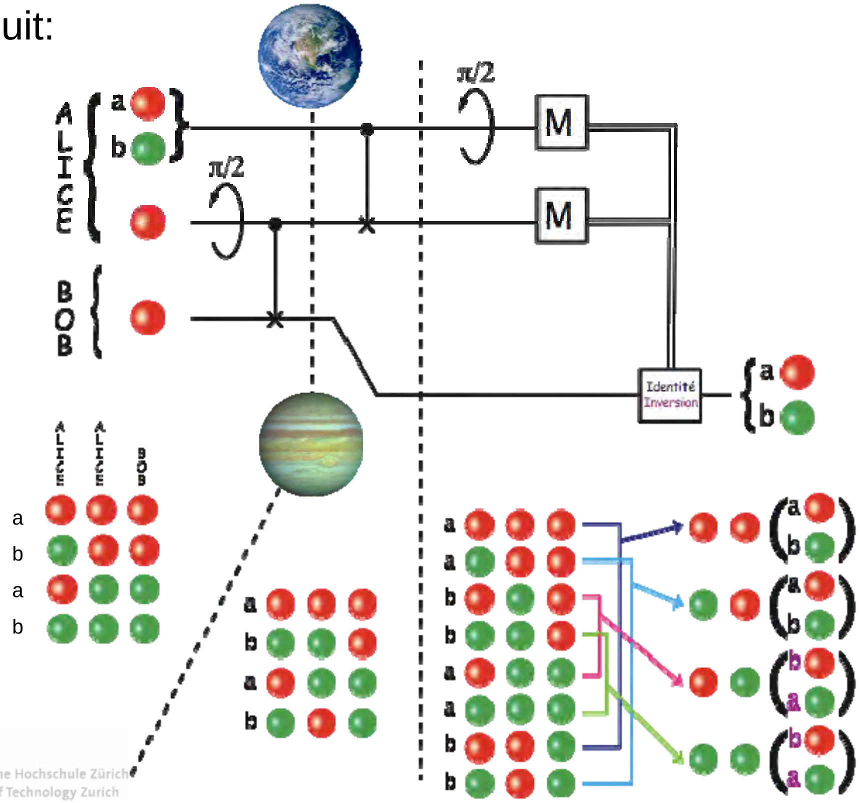
what could entanglement be used for?

teleportation of a quantum state

- a way for communicating an unknown quantum state
- a resource for quantum information processing

A scheme for teleportation

the circuit:



Quantum Parallelism

Classical: 2^n states for n classical bits

Quantum: simultaneous superposition of 2^n states

acting on 1 qubit has effect on the n-1 other qubits

1) simulation of quantum systems; Feynman 1982

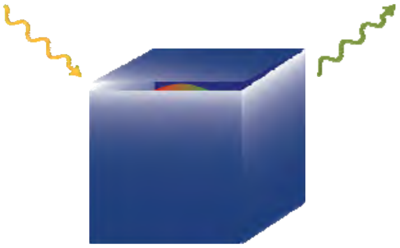
2) factoring; Shor 1994

3) searching; Grover 1996

Experimental Realization

Decoherence

superposition states are fragile: effect of the measurement

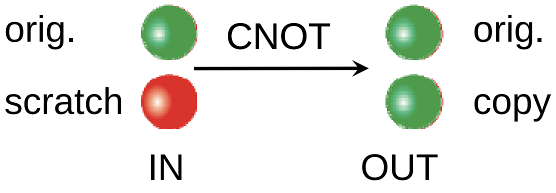


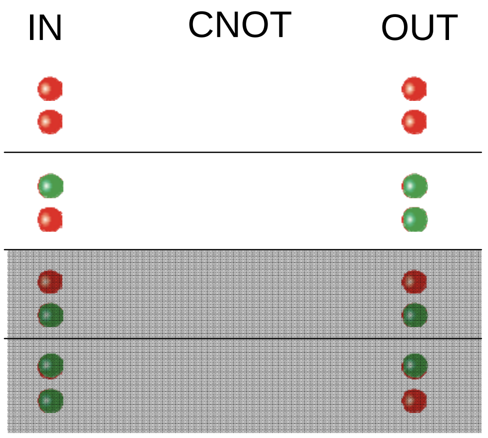
Required qubit properties:

- sensitive to external manipulation
- insensitive to rest of the world (environment)

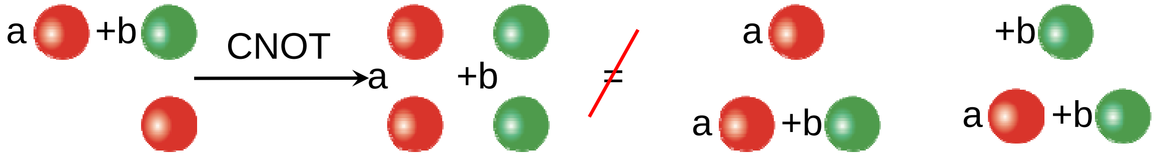
Copying (Qu)bits

classical copying circuit

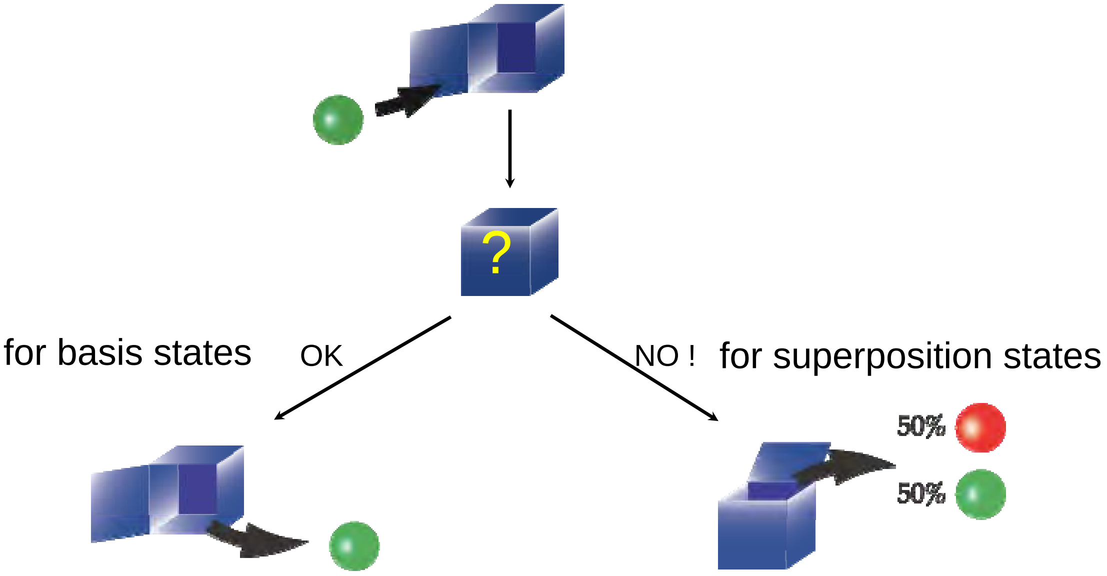

orig.
scratch
CNOT
orig.
copy
IN
OUT



quantum version

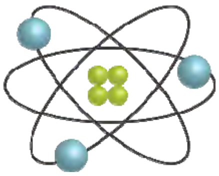


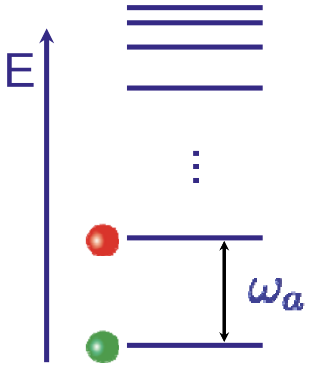
Copying qubits is impossible: No cloning theorem


for basis states
OK
NO !
for superposition states

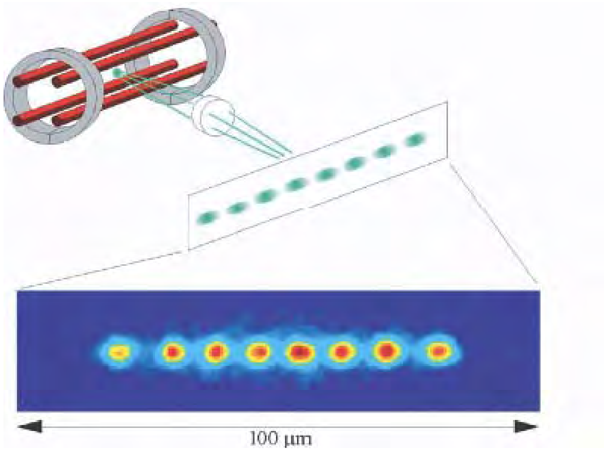
- additional difficulty to deal with errors in quantum information

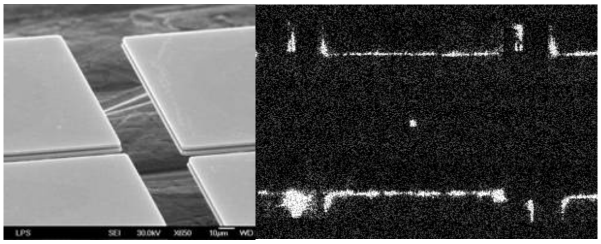
Experimental Realizations: Ions in Traps





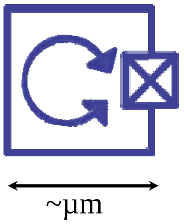
(non-linear)

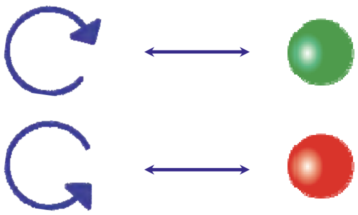


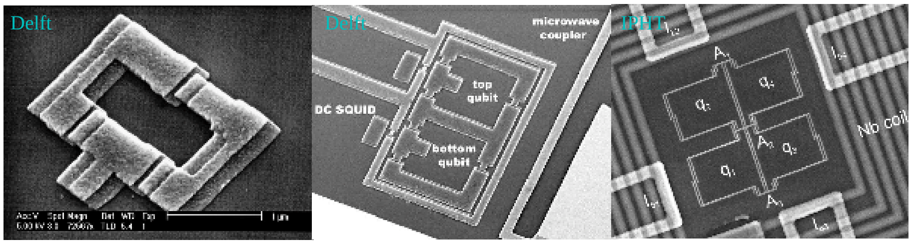


Superconducting Qubits

Josephson junction: non-linear inductance







coherent collective properties of billions of electrons

Summary

- Quantum information has interesting additional features in comparison to classical information
  - superposition of states
  - entanglement
- Promises to speed-up some important information processing tasks
- Experimental realizations are under development
  - ion traps
  - superconducting circuits
  - semiconductor quantum dots
  - nuclear spins

Making predictions about the future can be difficult

"Where a calculator on the Eniac is equipped with 18,000 vacuum tubes and weighs 30 tons, computers in the future may have only 1,000 vacuum tubes and perhaps weigh 1-1/2 tons"

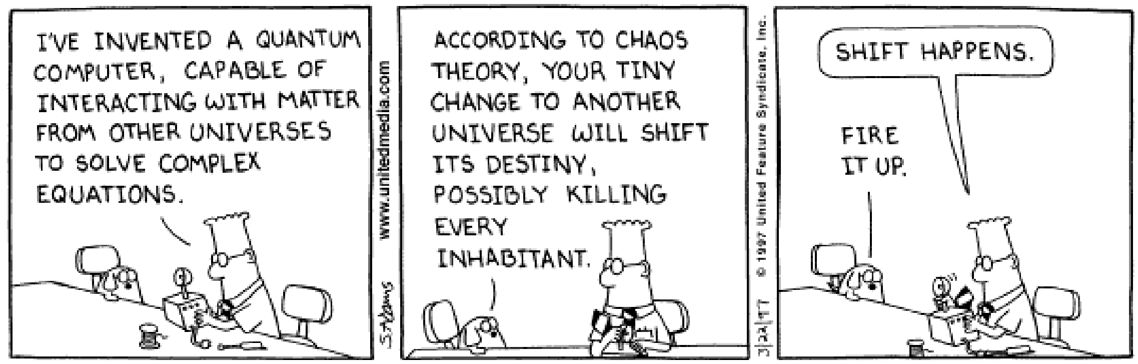
March, 1949 edition of Popular Mechanics

“I think there is a world market for about five computers”

remark attributed to Thomas J. Watson
(Chairman of the Board of IBM, 1943)

We are optimistic …

Dilbert's take on Quantum Computers:



… but careful too.

Quantum Bits

a **classical bit** (Binary digIT) can take values **either** 0 **or** 1

realized as a voltage level in a digital circuit (CMOS, TTL)

- 5 V = 1
- 0 V = 0

Quantum **bit**s (qubits) are quantum mechanical systems with two distinct quantum mechanical states. Qubits can be realized in a wide variety of physical systems displaying quantum mechanical properties. These include atoms, ions, electronic and nuclear magnetic moments, charges in quantum dots, charges and fluxes in superconducting circuits and many more. A suitable qubit should fulfill the DiVincenzo criteria (will be discussed in detail later).

a **qu**antum **bit** can take values (quantum mechanical states)

$|0\rangle$, $|1\rangle$ in Dirac notation

or both of them at the same time, i.e. it can be in a superposition of states (discussed later).

the qubit states are represented as vectors in a 2-dimensional Hilbert space (a complex vector space with inner product)

$$|0\rangle = \begin{pmatrix}1\\0\end{pmatrix} \; ; \; |1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$$

QM postulate: The quantum state of an isolated physical system is completely described by its state vector in a complex vector space with a inner product (a **Hilbert Space** that is). The state vector is a unit vector in that space.

inner product: is a function that takes two vectors $|v\rangle, |w\rangle$ from the Hilbert space and generates a complex number $z \in \mathbb{C}$

$$(|v\rangle, |w\rangle) = (\langle v|, |w\rangle) = \langle v|w\rangle$$

with

$$\langle v|w\rangle = \langle w|v\rangle^*$$

$$\langle v|v\rangle \geq 0$$

$$\langle v|\sum_i \lambda_i |w_i\rangle = \sum_i \lambda_i \langle v|w_i\rangle$$

definition of **inner product on an n-dimensional Hilbert space** $\mathbb{C}^n$

$$\langle v|w\rangle = \sum_i v_i^* w_i = (v_1,\ldots,v_n)^* \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$$

definition of the **outer product:**

$$|v\rangle\langle w| = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} (w_1 \ldots w_n)^* = \begin{pmatrix} v_1 w_1^* & \cdots & v_n w_1^* \\ \vdots & & \vdots \\ v_n w_1^* & \cdots & v_n w_n^* \end{pmatrix}$$

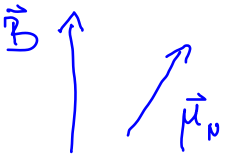
two state vectors are **orthogonal** when: $\langle v|w\rangle = 0$

the **norm** of a state vector is: $\| \, |v\rangle \| = \sqrt{\langle v|v\rangle}$

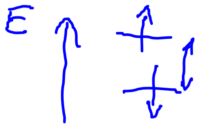
Physical Realizations of Qubits

nuclear spins in molecules:

- nuclear magnetic moment in external magnetic field



$$H_{int} = -\vec{\mu}_N \cdot \vec{B}$$



$\Delta E = g\,\mu_N B$

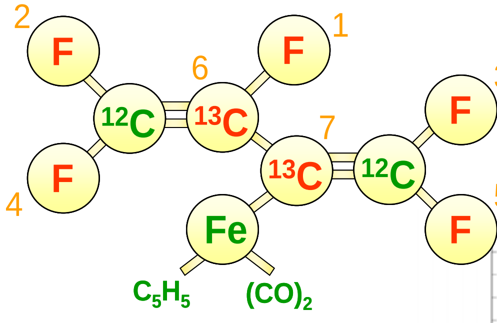
~ 450 MHz

~ 20 mK

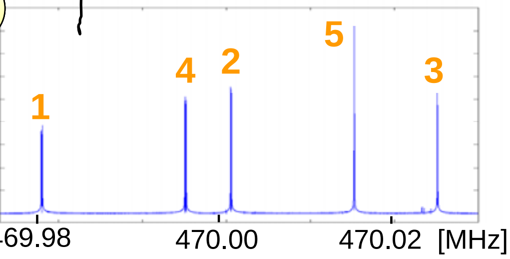
~ 2 μeV

$B \sim 10T$

- solution of large number of molecules with nuclear spin



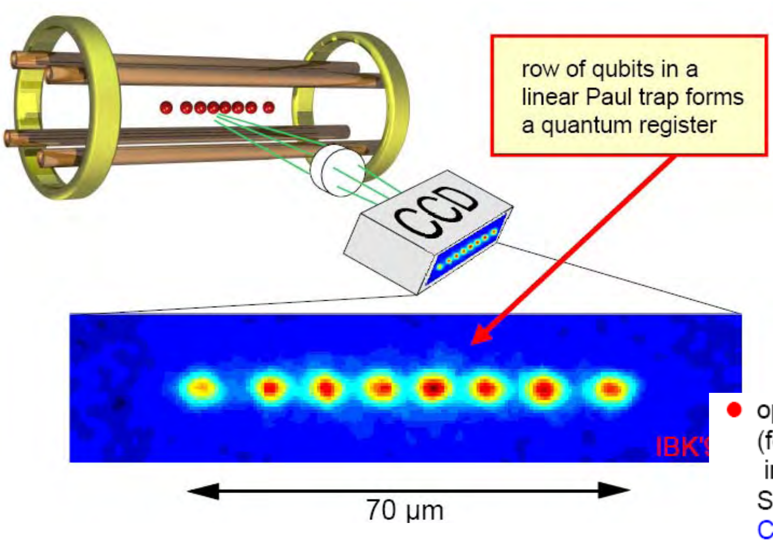
- distinct energies of different nuclei



figures from MIT group (www.mit.edu/~ichuang/)

$\longrightarrow E$

chain of **ions** in an ion trap:



row of qubits in a linear Paul trap forms a quantum register

- optical transition frequencies (forbidden transitions, intercombination lines)
S – D transitions in alkaline earths: Ca^+, Sr^+, Ba^+, Ra^+, (Yb^+, Hg^+) etc.

qubit states are implemented as long lived electronic states of atoms

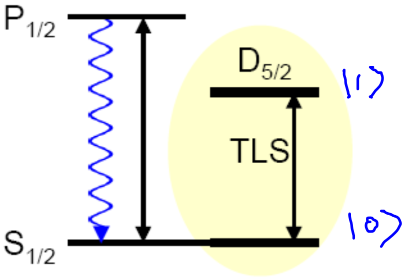
$\Delta E \sim 400$ THz

~ 20 kK

~ 2 eV

↑

$10^6 \times \Delta E$ for nuclei



figures from Innsbruck group
(http://heart-c704.uibk.ac.at/)

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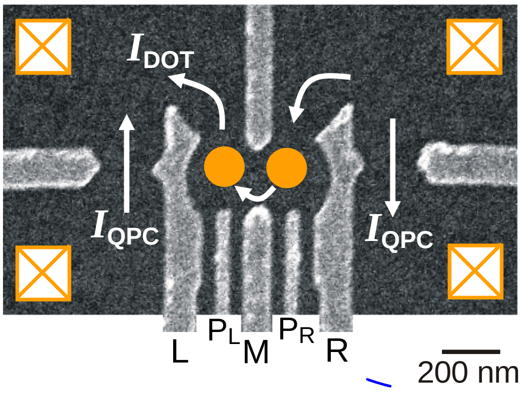
electrons in quantum dots:

- double quantum dot
- control individual electrons

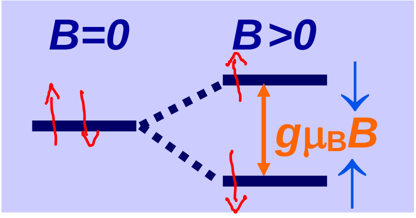


figures from Delft group
(http://qt.tn.tudelft.nl/)

NTT

GaAs/AlGaAs heterostructure
2DEG 90 nm deep
$n_s = 2.9 \times 10^{11}$ cm^{-2}

- spin states of electrons as qubit states
- interaction with external magnetic field B



↑ $B \sim 100$ mT

$\Delta E \sim 450$ MHz

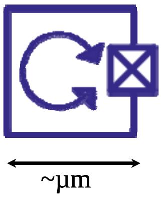
~ 20 mK

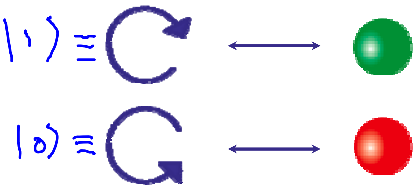
~ 2 μeV

superconducting circuits:

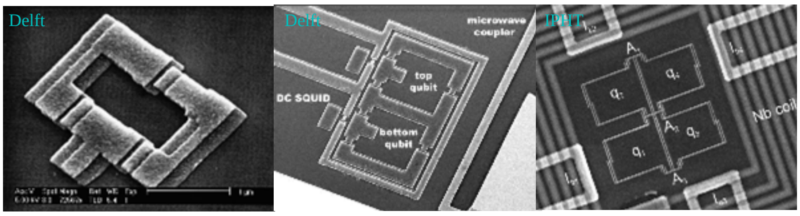
- qubits made from circuit elements

- circulating currents are qubit states





- made from sub-micron scale superconducting inductors and capacitors



$$\Delta E \sim 10\ \mathrm{GHz}$$
$$\sim 500\ \mathrm{mK}$$
$$\sim 50\ \mu\mathrm{eV}$$

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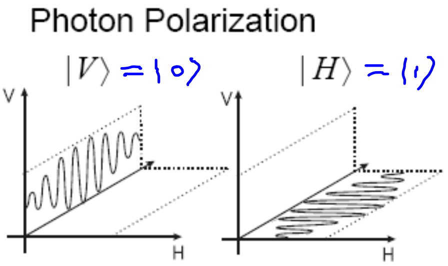
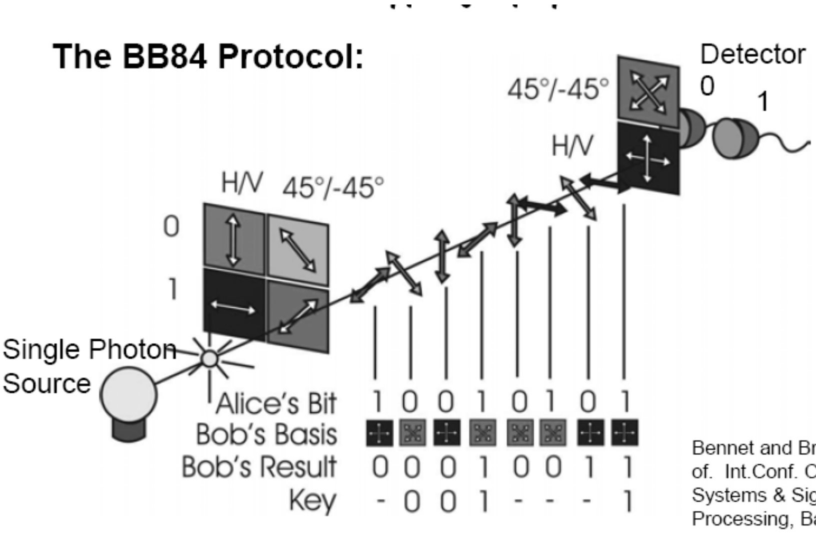
polarization states of photons:

- qubit states corresponding to different polarizations of a single photon (in the visible frequency range)



- are used in quantum cryptography and for quantum communication



Bennet and Brassard Proc of. Int.Conf. Computers, Systems & Signal Processing, Bangalore India 175 (1984) a qubit can be in a **superposition** of states

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle \quad \text{where} \quad \alpha, \beta \in \mathbb{C}$$

when the state of a qubit is measured one will find

$|0\rangle$ with probability $|\alpha|^2 = \alpha\,\alpha^*$

$|1\rangle$ '' $|\beta|^2 = \beta\,\beta^*$

where the normalization condition is $|\alpha|^2 + |\beta|^2 = 1$

this just means that the sum over the probabilities of finding the qubit in any state must be unity

example: $|\psi\rangle = \frac{1}{\sqrt{2}}\,|0\rangle + \frac{1}{\sqrt{2}}\,|1\rangle$

Bloch Sphere Representation of Qubit State Space

alternative representation of qubit state vector

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$$
$$= e^{i\gamma}\left[\cos\frac{\Theta}{2}\,|0\rangle + e^{i\varphi}\sin\frac{\Theta}{2}\,|1\rangle\right]$$

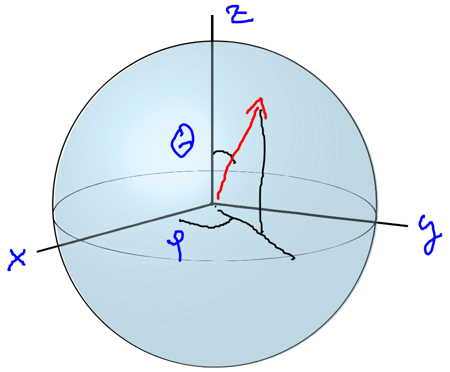
γ global phase factor

Θ azimuthal angle

φ polar angle

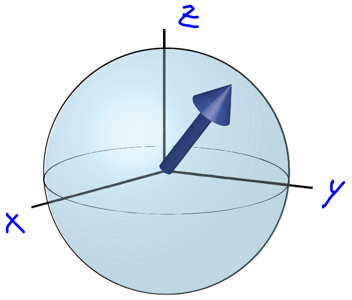
unit vector pointing at the surface of a sphere:

$$\vec{v} = (\cos\varphi\,\sin\Theta,\ \sin\varphi\,\sin\Theta,\ \cos\Theta)$$



- any single qubit operation can be represented as a rotation on a Bloch sphere (see exercises)
- there is no generalization of Bloch sphere picture to many qubits

information content in a single qubit:

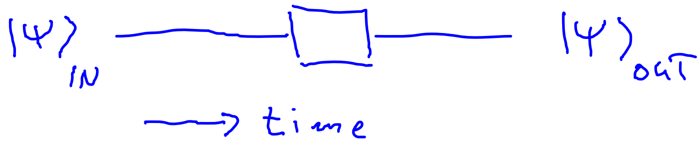


- infinite number of qubit states
- but single measurement reveals only 0 or 1 with probabilities $|\alpha|^2$ or $|\beta|^2$
- measurement will collapse state vector on basis state
- to determine α and β an infinite number of measurements has to be made

But, if not measured qubit contains 'hidden' information about α and β.

Single Qubit Gates

quantum circuit for a single qubit gate operation:



operations on single qubits:

$\boxed{X}$	bit flip	$\|0\rangle \longrightarrow \|1\rangle$; $\|1\rangle \longrightarrow \|0\rangle$
$\boxed{Y}$	bit flip*	$\|0\rangle \longrightarrow i\|1\rangle$; $\|1\rangle \longrightarrow -i\|0\rangle$
$\boxed{Z}$	phase flip	$\|0\rangle \longrightarrow \|0\rangle$; $\|1\rangle \longrightarrow -\|1\rangle$
$\boxed{I}$	identity	$\|0\rangle \longrightarrow \|0\rangle$; $\|1\rangle \longrightarrow \|1\rangle$

Pauli Matrices

The action of the single qubit gates discussed before can be represented by Pauli matrices acting on the computational basis states:

bit flip (NOT gate) $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$; $X|0\rangle = |1\rangle$; $X|1\rangle = |0\rangle$

bit flip* (with extra phase) $Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$; $Y|0\rangle = -i|1\rangle$; $Y|1\rangle = i|0\rangle$

phase flip $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$; $Z|0\rangle = |0\rangle$; $Z|1\rangle = -|1\rangle$

identity $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$; $I|0\rangle = |0\rangle$; $I|1\rangle = |1\rangle$

all are unitary: $U = X, Y, Z, I$: $U^\dagger U = I$

exercise: calculate eigenvalues and eigenvectors of all Pauli matrices and represent them on the Bloch sphere

---

Hadamard gate:

matrix representation of Hadamard gate:

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}} X + Z \qquad ; \quad H^\dagger H = I$$

generates superpositions of single qubit states

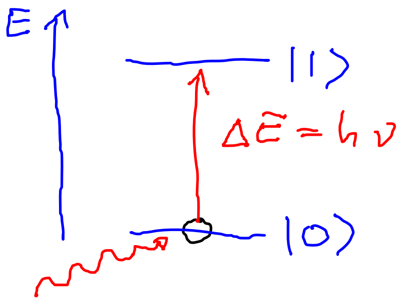
$$|0\rangle \;—\boxed{H}—\; \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$

$$|1\rangle \;—\boxed{H}—\; \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right)$$

exercise: write down the action of the Hadamard gate on the computational basis states of a qubit.

Control of Qubit States

by resonant irradiation:



using a pulse of radiation with controlled frequency ν amplitude A and length Δt

preparation of a superposition state:
assume the qubit to be in its ground state $|0\rangle$ initially

assume a pulse of length Δt will excite the qubit to state $|1\rangle$

then a pulse of length $\frac{\Delta t}{2}$ will bring the qubit to state $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

in fact such a pulse of chosen length and phase can prepare any single qubit state, i.e. any point on the Bloch sphere can be reached

---

Dynamics of a Quantum System:

QM postulate: The time evolution of a state ψ of a closed quantum system is described by a **Schrödinger equation**

$$i\hbar \frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle$$

where H is the hermitian operator known as the **Hamiltonian** describing the closed system.

a **closed quantum system** does not interact with any other system

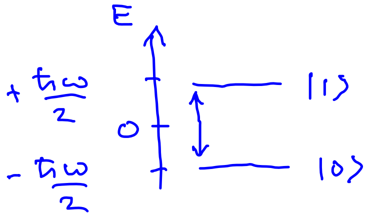
general solution: $$|\psi(t)\rangle = \exp\left[\frac{-iHt}{\hbar}\right]|\psi(0)\rangle$$

the **Hamiltonian**:

- H is hermitian and has a spectral decomposition
- with eigenvalues E
- and eigenvectors $|E\rangle$
- smallest value of E is the ground state energy with the eigenstate $|E\rangle$

$$H = \sum_E E|E\rangle\langle E|$$

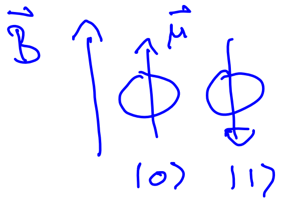
example:



$$H = -\frac{\hbar\omega}{2} Z$$

$$H = -\frac{\hbar\omega}{2} \left(|0\rangle\langle 0| - |1\rangle\langle 1| \right)$$

e.g. electron spin in a field:



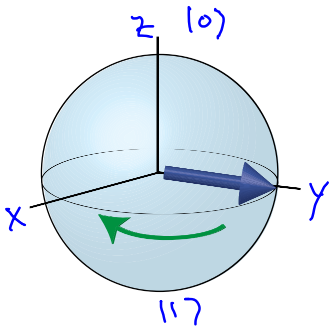
$$|\psi(0)\rangle = |0\rangle \longrightarrow |\psi(t)\rangle = e^{\frac{i\omega}{2}t} |0\rangle$$

$$|\psi(0)\rangle = |1\rangle \longrightarrow |\psi(t)\rangle = e^{-\frac{i\omega}{2}t} |1\rangle$$

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle + |1\rangle \right)$$

$$= \frac{1}{\sqrt{2}} \cancel{e^{i\frac{\omega}{2}t}} \left(|0\rangle + e^{-i\omega t} |1\rangle \right)$$

on the Bloch sphere:



$$|\psi\rangle = e^{i\gamma} \left(\cos\frac{\theta}{2} |0\rangle + e^{i\varphi} \sin\frac{\theta}{2} |1\rangle \right)$$

$$\Rightarrow \theta = \frac{\pi}{2}, \ \varphi = -\omega t$$

this is a rotation around the equator with Larmor precession frequency **ω**

---

generalized version:

QM postulate: The **evolution of a closed quantum system** is described by a unitary transformation U. That is the state $|\psi(t_2)\rangle$ is related to the state $|\psi(t_1)\rangle$ by a unitary operator that only depends on $t_2 - t_1$

$$|\psi(t_2)\rangle = U |\psi(t_1)\rangle$$

unitary operator (unitary matrix):

$$U^\dagger U = I$$

$$U^\dagger = (U^{tr})^*$$ hermitian conjugate

a unitary operator is a normal and hermitian operator

connection with Schroedinger equation:

- for any hermitian operator K the operator $\exp(iK)$ is unitary

normal operator: $A^\dagger A = A A^\dagger$

hermitian operator: $\langle v | A | w \rangle = \langle v | A^\dagger | w \rangle \; ; \; (A|v\rangle)^\dagger = \langle v | A^\dagger$

$\forall \; |v\rangle, |w\rangle \in V$ and a linear operator A

properties of unitary operators:

- preserve inner products $\langle v | U^\dagger U | w \rangle = \langle v | w \rangle$
- have a spectral decomposition $U = \sum_i u_i |u_i\rangle\langle u_i|$

u_i are eigenvalues of U

$|u_i\rangle$ are eigenvectors of U

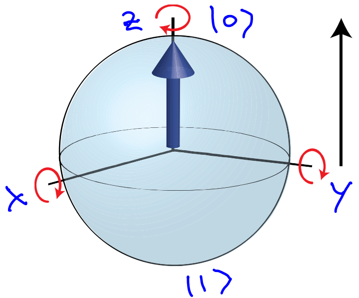
Rotation operators:

when exponentiated the Pauli matrices give rise to rotation matrices around the three orthogonal axis in 3-dimensional space.

$$R_x(\theta) = e^{-i\theta X/2} = \cos\frac{\theta}{2} I - i \sin\frac{\theta}{2} X = \begin{pmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$

$$R_y(\theta) = e^{-i\theta Y/2} = \cos\frac{\theta}{2} I - i \sin\frac{\theta}{2} Y = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$

$$R_z(\theta) = e^{-i\theta Z/2} = \cos\frac{\theta}{2} I - i \sin\frac{\theta}{2} Z = \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix}$$



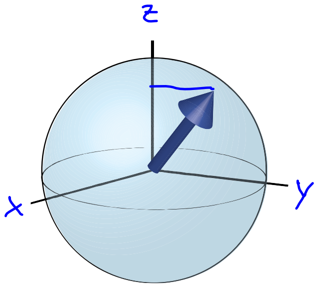
If the Pauli matrices X, Y or Z are present in the Hamiltonian of a system they will give rise to rotations of the qubit state vector around the respective axis.

exercise: convince yourself that the operators $R_{x,y,z}$ do perform rotations on the qubit state written in the Bloch sphere representation.

Quantum Measurement

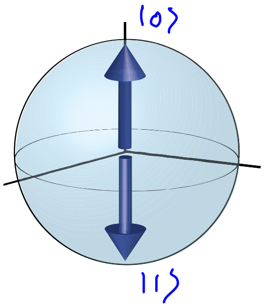
One way to determine the state of a qubit is to measure the projection of its state vector along a given axis, say the z-axis.

On the Bloch spehere this corresponds to the following operation:



After a projective measurement is completed the qubit will be in either one of its computational basis states.

In a repeated measurement the projected state will be measured with certainty.



QM postulate: quantum measurement is described by a set of operators $\{M_m\}$ acting on the state space of the system. The **probability** p **of a measurement result** m occurring when the state ψ is measured is

$$p(m) = \langle \psi | M_m^\dagger M_m | \psi \rangle$$

the **state of the system after the measurement** is

$$|\psi'\rangle = \frac{M_m |\psi\rangle}{\sqrt{p(m)}}$$

completeness: the sum over all measurement outcomes has to be unity

$$1 = \sum_m p(m) = \sum_m \langle \psi | M_m^\dagger M_m | \psi \rangle$$

example: projective measurement of a qubit in state ψ in its computational basis

$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$$

measurement operators:

$$|M_0\rangle = |0\rangle\langle 0| = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} \; ; \; M_1 = |1\rangle\langle 1| = \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$$

measurement probabilities:

$$p(0) = \langle \psi | M_0^\dagger M_0 | \psi \rangle = \alpha^* \alpha \langle 0|0\rangle = |\alpha|^2$$

$$p(1) = \langle \psi | M_1^\dagger M_1 | \psi \rangle = \beta^* \beta \langle 1|1\rangle = |\beta|^2$$

state after measurement:

$$\frac{M_0|\psi\rangle}{\sqrt{p(0)}} = \frac{\alpha|0\rangle}{\sqrt{|\alpha|^2}} = \frac{\alpha}{|\alpha|}|0\rangle$$

$$\frac{M_1|\psi\rangle}{\sqrt{p(1)}} = \frac{\beta|1\rangle}{\sqrt{|\beta|^2}} = \frac{\alpha}{|\alpha|}|1\rangle$$

measuring the state again after a first measurement yields the same state as the initial measurement with unit probability

---

Two Qubits:

2 classical bits with states:

bit 1	bit 2
0	0
0	1
1	0
1	1

- 2^n different states (here $n=2$)
- but only one is realized at any given time

2 qubits with quantum states:

qubit 1 qubit 2

$|00\rangle$

$|01\rangle$

$|10\rangle$

$|11\rangle$

- 2^n basis states ($n=2$)
- can be realized simultaneously
- quantum parallelism

2^n complex coefficients describe quantum state

$$|\psi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$$

normalization condition $$\sum_{ij} |\alpha_{ij}|^2 = 1$$

Composite quantum systems

QM postulate: The state space of a composite systems is the tensor product of the state spaces of the component physical systems. If the component systems have states ψ_i the composite system state is

$$|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \otimes \dots \otimes |\psi_n\rangle$$

This is a product state of the individual systems.

example:

$$|\psi_1\rangle = \alpha_1|0\rangle + \beta_1|1\rangle$$
$$|\psi_2\rangle = \alpha_2|0\rangle + \beta_2|1\rangle$$

$$\longrightarrow \quad |\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle = |\psi_1 \psi_2\rangle$$
$$= \alpha_1\alpha_2|00\rangle + \alpha_1\beta_2|01\rangle + \beta_1\alpha_2|10\rangle + \beta_1\beta_2|11\rangle$$

exercise: Write down the state vector (matrix representation) of two qubits, i.e. the tensor product, in the computational basis. Write down the basis vectors of the composite system.

Information content in multiple qubits

- 2^n complex coefficients describe state of a composite quantum system with n qubits!
- Imagine to have 500 qubits, then 2^{500} complex coefficients describe their state.
- How to store this state. 2^{500} is larger than the number of atoms in the universe. It is impossible in classical bits. This is also why it is hard to simulate quantum systems on classical computers.
- A quantum computer would be much more efficient than a classical computer at simulating quantum systems.
- Make use of the information that can be stored in qubits for quantum information processing!

Operators on composite systems:

Let A and B be operators on the component systems described by state vectors $|a>$ and $|b>$. Then the operator acting on the composite system is written as

$$A \otimes B \left(|a\rangle \otimes |b\rangle \right) = A|a\rangle \otimes B|b\rangle$$

tensor product in matrix representation (example for 2D Hilbert spaces):

$$A \otimes B = \begin{pmatrix} A_{11} B & A_{12} B \\ A_{21} B & A_{22} B \end{pmatrix}$$

$$|a\rangle \otimes |b\rangle = \begin{pmatrix} a_1 |b\rangle \\ a_2 |b\rangle \end{pmatrix} = \begin{pmatrix} a_1 b_1 \\ a_1 b_2 \\ a_2 b_1 \\ a_2 b_2 \end{pmatrix}$$

Entanglement:

Definition: An **entangled state** of a composite system is a state that cannot be written as a product state of the component systems.

example: an entangled 2-qubit state (one of the Bell states)

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left(|00\rangle + |11\rangle \right)$$

What is special about this state? Try to write it as a product state!

$$|\psi_1\rangle = \alpha_1 |0\rangle + \beta_1 |1\rangle \; ; \; |\psi_2\rangle = \alpha_2 |0\rangle + \beta_2 |1\rangle$$

$$|\psi_1 \psi_2\rangle = \alpha_1 \alpha_2 |00\rangle + \alpha_1 \beta_2 |01\rangle + \beta_1 \alpha_2 |10\rangle + \beta_1 \beta_2 |11\rangle$$

$$|\psi\rangle \overset{!}{=} |\psi_1 \psi_2\rangle \Rightarrow \alpha_1 \alpha_2 = \frac{1}{\sqrt{2}} \wedge \beta_1 \beta_2 = \frac{1}{\sqrt{2}} \Rightarrow \alpha_1 \beta_2 \neq 0$$

$$\wedge \alpha_2 \beta_1 \neq 0 !$$

It is not possible! This state is special, it is entangled!

Measurement of single qubits in an entangled state:

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$$

measurement of first qubit:

$$p_1(0) = \langle\psi|(M_0 \otimes I)^{+}(M_0 \otimes I)|\psi\rangle = \frac{1}{\sqrt{2}}\langle 00|\frac{1}{\sqrt{2}}|00\rangle = \frac{1}{2}$$

post measurement state:

$$|\psi'\rangle = \frac{(M_0 \otimes I)|\psi\rangle}{\sqrt{p_1(0)}} = \frac{\frac{1}{\sqrt{2}}|00\rangle}{\frac{1}{\sqrt{2}}} = |00\rangle$$

measurement of qubit two will then result with certainty in the same result:

$$P_2(0) = \langle\psi'|(I \otimes M_0)^{+}(I \otimes M_0)|\psi'\rangle = 1$$

The two measurement results are **correlated**! Correlations in quantum systems can be stronger than correlations in classical systems. This can be generally proven using the **Bell inequalities** which will be discussed later. Make use of such correlations as a **resource** for information processing, for example in **super dense coding** and **teleportation**.

Super Dense Coding

task: Try to transmit two bits of classical information between you (Bob) and your friend Alice (A) using only one qubit. (As you are living in a quantum world you are allowed to use on pair of entangled qubits that you have prepared ahead of time.)

protocol:

A) Alice and Bob each have one qubit of an entangled pair in their possession

$$|\psi\rangle = 1/\sqrt{2}\,\left(|00\rangle + |11\rangle\right)$$

B) Bob does a quantum operation on his qubit depending on which 2 classical bits he wants to communicate

C) Bob sends his qubit to Alice

D) Alice does one measurement on the entangled pair

shared entanglement $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$

local operations $X_1 ; Y_1 ; Z_1 ; I_1$

send Bobs qubit to Alice

Alice measures

bits to be transferred:	Bobs operation	resulting 2-qubit state	Alice's operation
00	I_1	$I_1 \lvert\psi\rangle = \frac{1}{\sqrt{2}}(\lvert 00\rangle + \lvert 11\rangle)$	measure in Bell basis
01	Z_1	$Z_1 \lvert\psi\rangle = \frac{1}{\sqrt{2}}(\lvert 00\rangle - \lvert 11\rangle)$	
1 0	X_1	$X_1 \lvert\psi\rangle = \frac{1}{\sqrt{2}}(\lvert 10\rangle + \lvert 01\rangle)$	
11	iY_1	$iY_1 \lvert\psi\rangle = \frac{1}{\sqrt{2}}(\lvert 10\rangle - \lvert 01\rangle)$	

- all these states are entangled (try!)
- they are called the Bell states

comments:

- two qubits are involved in protocol BUT Bob only interacts with one and sends only one along his quantum communications channel
- two bits cannot be communicated sending a single classical bit along a classical communications channel

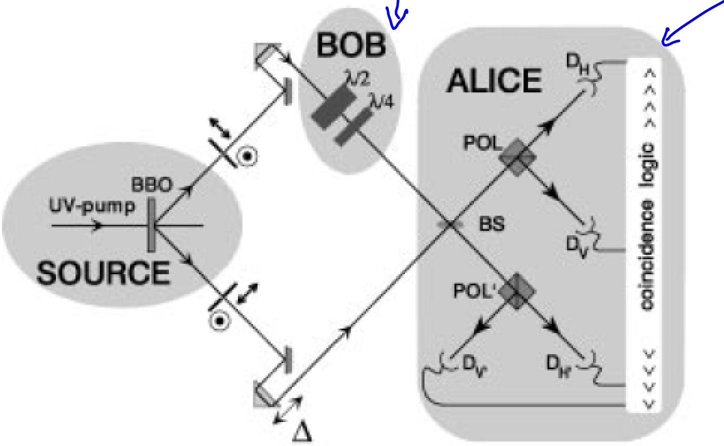
proposal of super dense coding and experimental demonstration using photons:

Communication via one- and two-particle operators on Einstein-Podolsky-Rosen states
Phys. Rev. Lett.69, 2881(1992)
Charles H. Bennett and Stephen J. Wiesner

Dense coding in experimental quantum communication
Phys. Rev. Lett.76, 4656 (1996)
Klaus Mattle, Harald Weinfurter, Paul G. Kwiat, and Anton Zeilinger

state manipulation

Bell state measurement



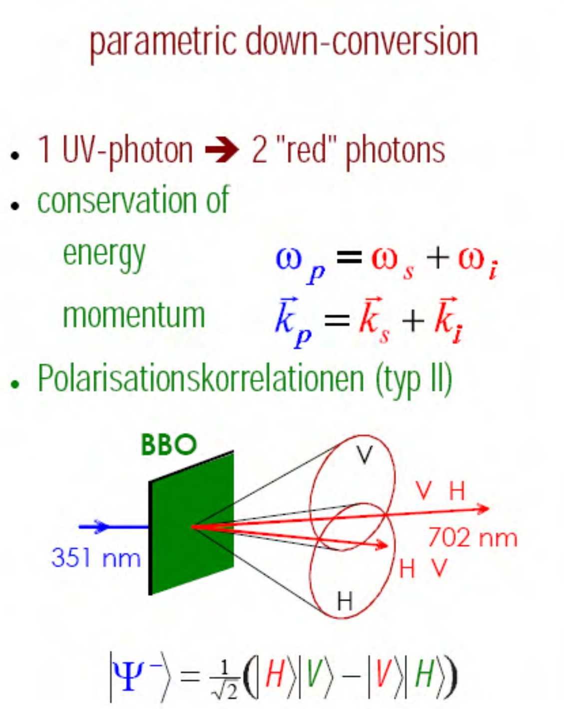
$$\psi^- = \frac{1}{\sqrt{2}}(|HV\rangle - |VH\rangle) \quad \text{asym.}$$

$$\left.\begin{aligned} \psi^+ &= \frac{1}{\sqrt{2}}(|HV\rangle + |VH\rangle) \\ \phi^+ &= \frac{1}{\sqrt{2}}(|HH\rangle + |VV\rangle) \\ \phi^- &= \frac{1}{\sqrt{2}}(|HH\rangle - |VV\rangle) \end{aligned}\right\} \text{sym.}$$

H = horizontal polarization
V = vertical polarization

Parametric Down Conversion:

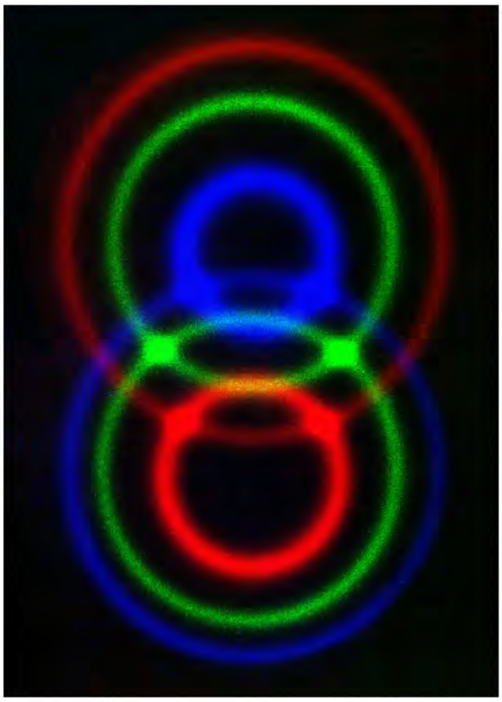
a source of polarization entangled photon pairs



parametric down-conversion

- 1 UV-photon ➔ 2 "red" photons
- conservation of
  - energy $\omega_p = \omega_s + \omega_i$
  - momentum $\vec{k}_p = \vec{k}_s + \vec{k}_i$
- Polarisationskorrelationen (typ II)

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|H\rangle|V\rangle - |V\rangle|H\rangle)$$



Classical Logic Gates:

		IN	OUT
non trivial single bit logic gate:	NOT	0	1
		1	0
universal two bit logic gate: AND followed by NOT	NAND	00	1
		01	1
		10	1
		11	0

Other gates exist (AND, OR, XOR, NOR) but can all be implemented using NAND gates.

universality of NAND: Any function operating on bits can be computed using NAND gates. Therefore NAND is called a universal gate.

Two Qubit Quantum Logic Gates

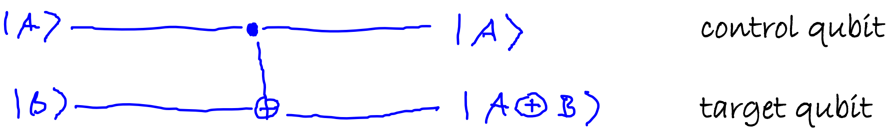
The **controlled NOT gate** (CNOT):

function:

$$|00\rangle \longrightarrow |00\rangle$$
$$|01\rangle \longrightarrow |01\rangle$$
$$|10\rangle \longrightarrow |11\rangle$$
$$|11\rangle \longrightarrow |10\rangle$$

$$|A,B\rangle \longrightarrow |A, A\oplus B\rangle$$ addition mod 2 of basis states

CNOT circuit:

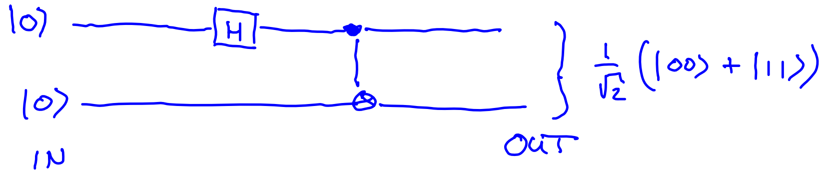


comparison with classical gates:

- XOR is not reversible
- CNOT is reversible (unitary)

Universality of controlled NOT:
Any multi qubit logic gate can be composed of CNOT gates and single qubit gates X,Y,Z.

application of CNOT: **generation of entangled states (Bell states)**:



$$|00\rangle \xrightarrow{H_1} \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$|01\rangle \xrightarrow{H_1} \frac{1}{\sqrt{2}}(|01\rangle + |11\rangle) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$

$$|10\rangle \xrightarrow{H_1} \frac{1}{\sqrt{2}}(|00\rangle - |10\rangle) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$

$$|11\rangle \xrightarrow{H_1} \frac{1}{\sqrt{2}}(|01\rangle - |11\rangle) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

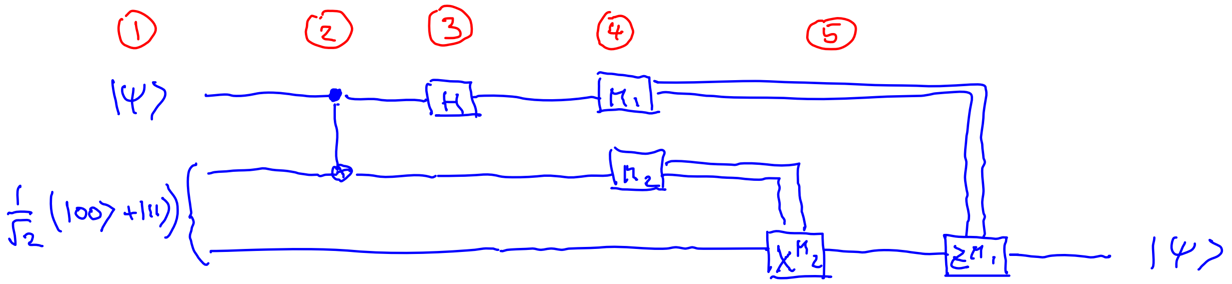
exercise: Write down the unitary matrix representations of the CNOT in the computational basis with qubit 1 being the control qubit. Write down the matrix in the same basis with qubit 2 being the control bit.

Quantum Teleportation:

Task: Alice wants to transfer an unknown quantum state ψ to Bob only using **one entangled pair** of qubits and **classical information** as a resource.

note:
- Alice does not know the state to be transmitted
- Even if she knew it the classical amount of information that she would need to send would be infinite.

The **teleportation circuit**:



original article:

Teleporting an unknown quantum state via dual classical and Einstein-Podolsky-Rosen channels
Charles H. Bennett, Gilles Brassard, Claude Crépeau, Richard Jozsa, Asher Peres, and William K. Wootters
Phys. Rev. Lett. **70**, 1895 (1993) [PROLA Link]

How does it work?

① $|\psi\rangle \otimes \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}(\alpha|000\rangle + \alpha|011\rangle + \beta|100\rangle + \beta|111\rangle)$

CNOT between qubit to be teleported and one bit of the entangled pair:

② $\xrightarrow{CNOT_{12}} \frac{1}{\sqrt{2}}(\alpha|000\rangle + \alpha|011\rangle + \beta|110\rangle + \beta|101\rangle)$

Hadamard on qubit to be teleported:

③ $\xrightarrow{H_1} \frac{1}{2}\Big[(|00\rangle(\alpha|0\rangle + \beta|1\rangle) + |10\rangle(\alpha|0\rangle - \beta|1\rangle)$

$+ |01\rangle(\alpha|1\rangle + \beta|0\rangle) + |11\rangle(\alpha|1\rangle - \beta|0\rangle)\Big]$

measurement of qubit 1 and 2, classical information transfer and single bit manipulation on target qubit 3:

⑤

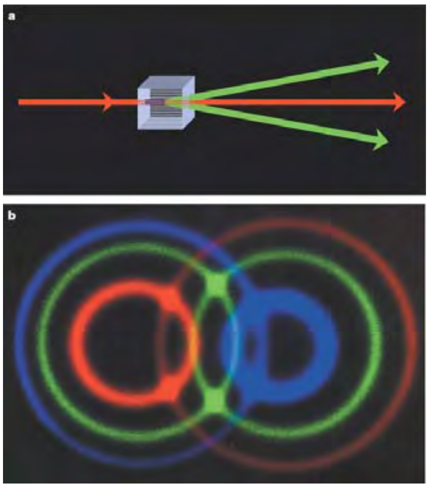
④ $\xrightarrow{M_1 \otimes M_2}$ $P_{00} = \frac{1}{4}$; $|\psi_3\rangle = \alpha|0\rangle + \beta|1\rangle \xrightarrow{I} |\psi\rangle$

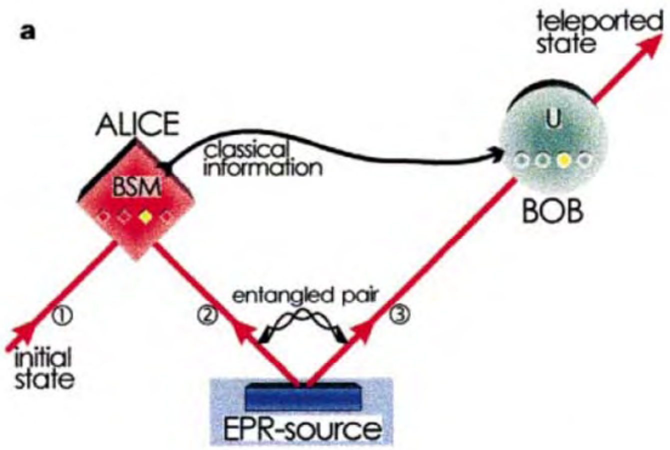
$P_{10} = \frac{1}{4}$; $|\psi_3\rangle = \alpha|0\rangle - \beta|1\rangle \xrightarrow{Z} |\psi\rangle$

$P_{01} = \frac{1}{4}$; $|\psi_3\rangle = \alpha|1\rangle + \beta|0\rangle \xrightarrow{X} |\psi\rangle$

$P_{11} = \frac{1}{4}$; $|\psi_3\rangle = \alpha|1\rangle - \beta|0\rangle \xrightarrow{XZ} |\psi\rangle$

(One) Experimental Realization of Teleportation using Photon Polarization:





- parametric down conversion (PDC) source of entangled photons
- qubits are polarization encoded

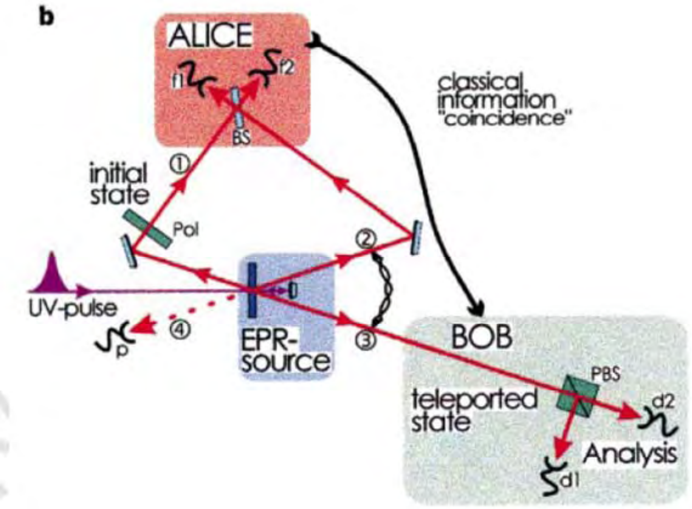
Experimental quantum teleportation
Dik Bouwmeester, Jian-Wei Pan, Klaus Mattle, Manfred Eibl, Harald Weinfurter, Anton Zeilinger
Nature 390, 575 - 579 (11 Dec 1997) Article
Abstract | Full Text | PDF | Rights and permissions | Save this link

Experimental Implementation

start with states

$$|\psi_1\rangle = \alpha |H\rangle + \beta |V\rangle$$

$$|\psi_{23}\rangle = \frac{1}{\sqrt{2}} \left(|HV\rangle - |VH\rangle \right)$$



combine photon to be teleported (1) and one photon of entangled pair (2) on a 50/50 beam splitter (BS) and measure (at Alice) resulting state in Bell basis.

analyze resulting teleported state of photon (3) using polarizing beam splitters (PBS) single photon detectors

- polarizing beam splitters (PBS) as detectors of teleported states

teleportation papers for you to present:

Experimental Realization of Teleporting an Unknown Pure Quantum State via Dual Classical and Einstein-Podolsky-Rosen Channels
D. Boschi, S. Branca, F. De Martini, L. Hardy, and S. Popescu
Phys. Rev. Lett. **80**, 1121 (1998) [PROLA Link]

Unconditional Quantum Teleportation
A. Furusawa, J. L. Sørensen, S. L. Braunstein, C. A. Fuchs, H. J. Kimble, and E. S. Polzik
Science 23 October 1998 282: 706-709 [DOI: 10.1126/science.282.5389.706] (in Research Articles)
Abstract » Full Text » PDF »

Complete quantum teleportation using nuclear magnetic resonance
M. A. Nielsen, E. Knill, R. Laflamme
Nature 396, 52 - 55 (05 Nov 1998) Letters to Editor
Abstract | Full Text | PDF | Rights and permissions | Save this link

Deterministic quantum teleportation of atomic qubits
M. D. Barrett, J. Chiaverini, T. Schaetz, J. Britton, W. M. Itano, J. D. Jost, E. Knill, C. Langer, D. Leibfried, R. Ozeri, D. J. Wineland
Nature 429, 737 - 739 (17 Jun 2004) Letters to Editor
Abstract | Full Text | PDF | Rights and permissions | Save this link

Deterministic quantum teleportation with atoms
M. Riebe, H. HÃ¤ffner, C. F. Roos, W. HÃ¤nsel, J. Benhelm, G. P. T. Lancaster, T. W. KÃ¶rber, C. Becher, F. Schmidt-Kaler, D. F. V. James, R. Blatt
Nature 429, 734 - 737 (17 Jun 2004) Letters to Editor
Abstract | Full Text | PDF | Rights and permissions | Save this link

Quantum teleportation between light and matter
Jacob F. Sherson, Hanna Krauter, Rasmus K. Olsson, Brian Julsgaard, Klemens Hammerer, Ignacio Cirac, Eugene S. Polzik
Nature 443, 557 - 560 (05 Oct 2006) Letters to Editor
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John Bell's thought experiment

- Charlie simultaneously prepares two particles having physical properties Q, R, S, T and gives one particle each to Alice and Bob.
- Alice measures the properties Q and R of her particle with the possible outcomes $q = \pm 1$ and $r = \pm 1$.
- Bob simultaneously measures the properties S and T of his particles with the possible outcomes $s = \pm 1$ and $t = \pm 1$.

consider the quantity: $QS + RS + RT - QT$

$$= (R+Q)S + (R-Q)T = \pm 2$$

since $R+Q = 0$
or $R-Q = 0$

the probability of the system being in state $Q=q,\ R=r,\ S=s,\ T=t$

is given by: $p(q,r,s,t)$

we also denote $E(X)$ as the mean of the quantity X : $E(x) = \sum_{q,r,s,t} p(q,r,s,t)\, x$

Now, Alice and Bob perform measurements on the two particles and record their outcomes. Then they meet up and perform the multiplications (e.g. $q\,s$) and calculate the average values $E(QS)$.

What are the possible outcomes of measuring the quantity $E(QS+RS+RT-QT)$?

find an upper bound:

$$E(QS+RS+RT-QT) = \sum_{q,r,s,t} p(q,r,s,t) \underbrace{(qs+rs+rt-qt)}_{< 2}$$

$$\le \underbrace{\sum_{q,r,s,t} p(q,r,s,t)}_{=1}\, 2$$

$$= 2$$

also:

$$\boxed{E(QS+RS+RT-QT)} = \sum_{q,r,s,t} p(q,r,s,t)\,(qs+rs+rt-qt)$$

$$= E(QS) + E(RS) + E(RT) - E(QT)$$

$$\boxed{\le 2}$$

Bell inequality

measure this quantity for a Bell state: $|\psi\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right)$

Alice measures: $Q = Z_1$ Bob measures: $S = \frac{1}{\sqrt{2}}\left(-Z_2 - X_2\right)$

$P = X_1$ $T = \frac{1}{\sqrt{2}}\left(Z_2 - X_2\right)$

determine expectation values of joint measurements:

$$\langle QS\rangle = \langle\psi|\; Z_1 \frac{1}{\sqrt{2}}\left(-Z_2 - X_2\right)|\psi\rangle = \frac{1}{\sqrt{2}}\;\langle\psi| -Z_1Z_2 - Z_1X_2|\psi\rangle$$

$$= \frac{1}{\sqrt{2}}\frac{1}{2}\left(\langle 01| - \langle 10|\right)\left[\left(|01\rangle - |10\rangle\right) - \left(|00\rangle + |11\rangle\right)\right] = \frac{1}{\sqrt{2}}\frac{1}{2}\left(1+1\right)$$

$$= \frac{1}{\sqrt{2}}$$

$$\langle RS\rangle = \frac{1}{\sqrt{2}} \;;\; \langle RT\rangle = \frac{1}{\sqrt{2}} \;;\; \langle QT\rangle = -\frac{1}{\sqrt{2}}$$

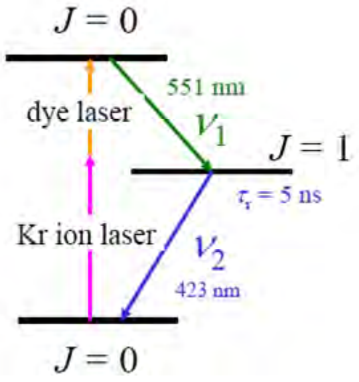
determine value of Bell inequality:

$$\langle QS\rangle + \langle RS\rangle + \langle RT\rangle - \langle QT\rangle = 2\sqrt{2} > 2 \quad !!!$$

Bell states maximally violate the Bell inequality!

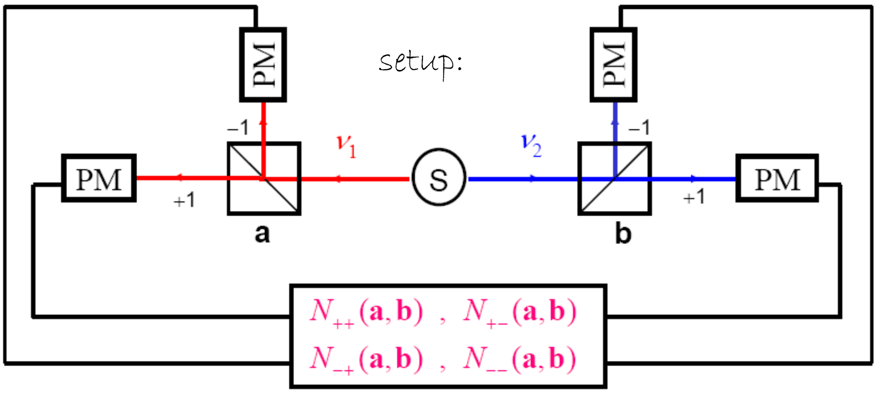
Experimental violation of Bell Inequality (Alain Aspect):

generation of polarization entangled photons: $\left|\Psi(\nu_1,\nu_2)\right\rangle = \frac{1}{\sqrt{2}}\left\{\left|x,x\right\rangle + \left|y,y\right\rangle\right\}$



$4p^2\,{}^1S_0 - 4s4p\,{}^1P_1 - 4p^2\,{}^1S_0$

radiative cascade in calcium 40



Experimental Realization of Einstein-Podolsky-Rosen-Bohm Gedankenexperiment: A New Violation of Bell's Inequalities
A. Aspect, P. Grangier, and G. Roger
Phys. Rev. Lett. 49, 91-94 (1982)
[PDF (682 kB)]

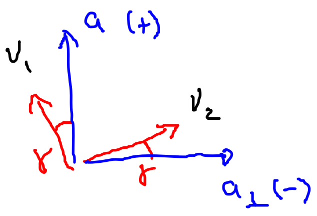
Experimental Tests of Realistic Local Theories via Bell's Theorem
A. Aspect, P. Grangier, and G. Roger
Phys. Rev. Lett. 47, 460-463 (1981)
[PDF (665 kB)]

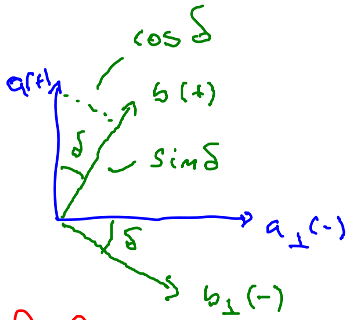
measure coincidences and calculate correlation coefficient:

$$\boxed{E(\mathbf{a},\mathbf{b}) = \frac{N_{++}(\mathbf{a},\mathbf{b}) - N_{+-}(\mathbf{a},\mathbf{b}) - N_{-+}(\mathbf{a},\mathbf{b}) + N_{--}(\mathbf{a},\mathbf{b})}{N_{++}(\mathbf{a},\mathbf{b}) + N_{+-}(\mathbf{a},\mathbf{b}) + N_{-+}(\mathbf{a},\mathbf{b}) + N_{--}(\mathbf{a},\mathbf{b})}}$$

if (a,b)=0 (parallel polarizers) then E(a,b) = 1, i.e. perfect correlation of results quantum mechanical prediction:

$$|\Psi(\nu_1,\nu_2)\rangle = \frac{1}{\sqrt{2}}\{|x,x\rangle + |y,y\rangle\}$$





probability of individual photon measurements

$$P_+(\mathbf{a}) = P_-(\mathbf{a}) = \frac{1}{2} \quad ; \quad P_+(\mathbf{b}) = P_-(\mathbf{b}) = \frac{1}{2}$$

$$\frac{1}{2}|(\cos^2\gamma + \sin^2\gamma) = \frac{1}{2} = P_+ = P_-$$

for any γ

probabilities of joint measurements on both photons:

$$P_{++}(\mathbf{a},\mathbf{b}) = P_{--}(\mathbf{a},\mathbf{b}) = \frac{1}{2}\cos^2(\mathbf{a},\mathbf{b})$$

$$P_{+-}(\mathbf{a},\mathbf{b}) = P_{-+}(\mathbf{a},\mathbf{b}) = \frac{1}{2}\sin^2(\mathbf{a},\mathbf{b})$$

easy to see for $\gamma = 0$

$$E(\mathbf{a},\mathbf{b}) = P_{++} + P_{--} - P_{+-} - P_{-+}$$

$$E_{\text{MQ}}(\mathbf{a},\mathbf{b}) = \cos 2(\mathbf{a},\mathbf{b}) = \cos^2(a,b) - \sin^2(a,b)$$

$$= \frac{1}{\sqrt{2}} \quad \text{for } (a,b) = \frac{\pi}{8}$$

---

measure Bell inequality:

$$S = E(\mathbf{a},\mathbf{b}) - E(\mathbf{a},\mathbf{b}') + E(\mathbf{a}',\mathbf{b}) + E(\mathbf{a}',\mathbf{b}')$$

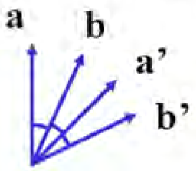
with:

$$\boxed{E(\mathbf{a},\mathbf{b}) = \frac{N_{++}(\mathbf{a},\mathbf{b}) - N_{+-}(\mathbf{a},\mathbf{b}) - N_{-+}(\mathbf{a},\mathbf{b}) + N_{--}(\mathbf{a},\mathbf{b})}{N_{++}(\mathbf{a},\mathbf{b}) + N_{+-}(\mathbf{a},\mathbf{b}) + N_{-+}(\mathbf{a},\mathbf{b}) + N_{--}(\mathbf{a},\mathbf{b})}}$$

repeat for different angles between polarizer (a,b) = θ:

$$(\mathbf{a},\mathbf{b}) = (\mathbf{b},\mathbf{a}') = (\mathbf{a}',\mathbf{b}) = \frac{\pi}{8}$$

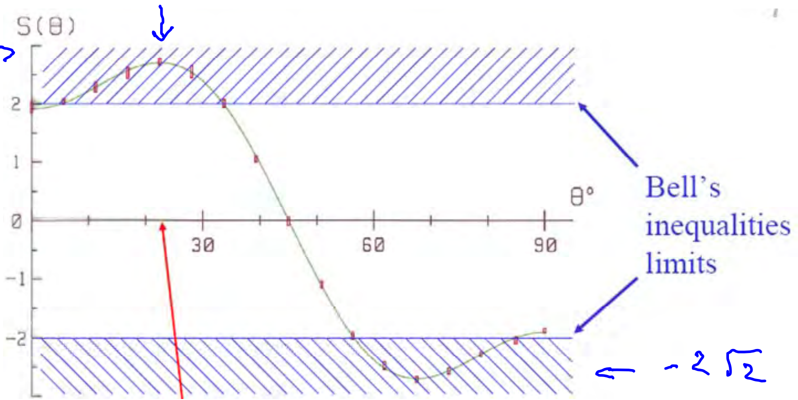
= 22.5 deg



experimental result:

$\pi/8 = 22.5$ deg

$2\sqrt{2}$ →



← $-2\sqrt{2}$

$$\frac{\pi}{4} + \frac{\pi}{8} = 67.5 \text{ deg}$$

comments:

Consequences of violation of Bell inequalities:

- The assumption that physical properties (e.g. Q, R, S, T) of systems have values which exists independent of observation (the **Realism Assumption**) is wrong.
- The assumption that experiments performed at one point in time and space (at Alices) cannot be influenced by experiments at another point in time and space (at Bobs, in a different light cone) (the **Locality Assumption**) is wrong.

Both of the above assumptions are sometimes called **Local Realism**.

Quantum mechanics violates these assumptions, as shown in experiments!

Test of Locality: The Innsbruck Experiment

Violation of Bell's Inequality under Strict Einstein Locality Conditions
G. Weihs, T. Jennewein, C. Simon, H. Weinfurter, and A. Zeilinger
Phys. Rev. Lett. 81, 5039-5043 (1998)
[PDF (195 kB)]

Reversible Classical Logic Gates

irreversible computation: Information is erased. E.g. in the standard AND gate the input bits are lost and cannot be reconstructed from the output.

Landauer's Principle: When a computer erases a single bit of information the amount of **energy dissipated** in the environment is at least $k_B\ T \ln 2$, where k_B is the Boltzmann constant and T is the temperature of the environment. (Equivalent statement: When a computer erases a single bit of information the **entropy** of the environment **increases** by at least $k_B \ln 2$.)

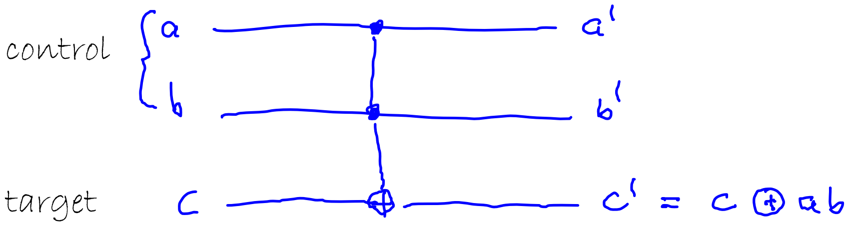
note: Today's computers dissipate about $500\ k_B\ T \ln 2$ per elementary logic operation.

reversible computation: No information is ever erased. The input can always be reconstructed from the output

consequence: Reversible computation (e.g. quantum computation) can (in principle) be done without energy dissipation. (However, you may have to reset your memory at some point dissipating energy. Or you may perform a read-out (measurement) of the result of the computation that may dissipate energy.)

The Toffoli gate

circuit representation:



truth table:

flip target bit c if and only if control bits **a** and **b** are 1.

abc	a'b'c'
000	000
001	001
010	010
011	011
100	100
101	101
110	111
111	110

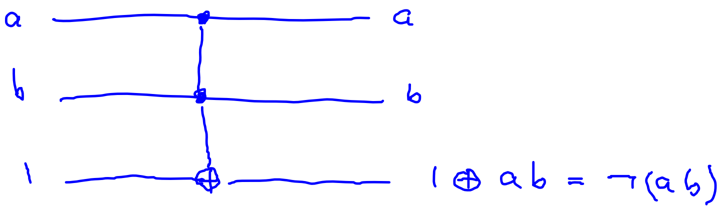
properties:

- this gate is reversible
- it can be used to simulate classical NAND and FANOUT gates in a reversible way
- an arbitrary circuit can be simulated efficiently using Toffoli gates

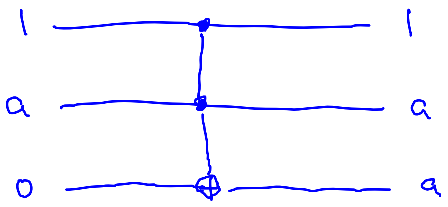
---

Simulating classical gates using the Toffoli gate and **ancilla** bits (but possibly leaving some unused garbage bits behind)

Simulation of NAND



Simulation of FANOUT

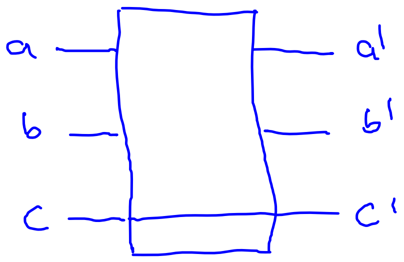


The Fredkin Gate

is a universal and reversible logic gate

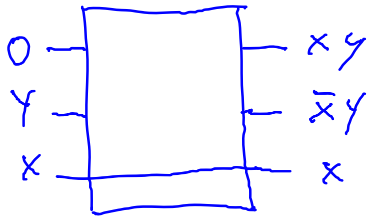
truth table:
swap target bits **a** and **b** if
control bit **c** is 1.

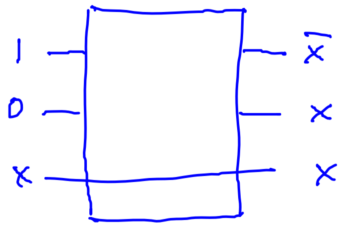
abc	a'b'c'
000	000
001	001
010	010
011	101
100	100
101	011
110	110
111	111

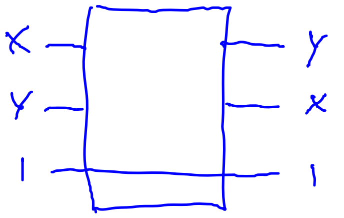


this gate is **conservative**, i.e. the number of 1s at the input and output are conserved

Simulating classical logic gates reversibly:


AND


NOT


CROSSOVER

---

properties:

- ancilla bits prepared in special states (0, 1) are allowed at input
- extraneous (garbage) bits remain at output

How to control the states of ancilla bits? How to avoid that the final state of some ancilla bits contain information that depends on the input state of the system? (The availability of such information will destroy quantum interference effects that we want to use for quantum information processing.)

general function evaluation

$$(x,0) \rightarrow (\overset{\text{evaluated function}}{f(x)}, \overset{\text{garbage bits}}{g(x)})$$

Make use of classical NOT and CNOT gates to start out with all ancilla bits in state 0 and to reset extraneous states to some standard state that does not depend on the input after the function evaluation.

remember: Classical version of CNOT can be used to copy classical bits from control to target state if target state is initialized to 0.

$$\text{CNOT} \equiv (c, t \oplus c)$$

constructing a reversible version of the function *f*:

initial state: $(\overset{\text{input}}{x}, \overbrace{0, 0}^{\text{ancillas}}, \overset{\text{output}}{y})$

make a copy of input bit string x: $\xrightarrow{CNOT_{12}} (x, x, 0, y)$

compute f: $\xrightarrow{f} (x, f(x), g(x), y)$

bitwise addition of f(x) to y using CNOT: $\xrightarrow{CNOT_{24}} (x, f(x), g(x), y \oplus f(x))$

run *f* backwards (it is reversible): $\xrightarrow{f^{-1}} (x, 0, 0, y \oplus f(x))$

expression for **general reversible function evaluation**:

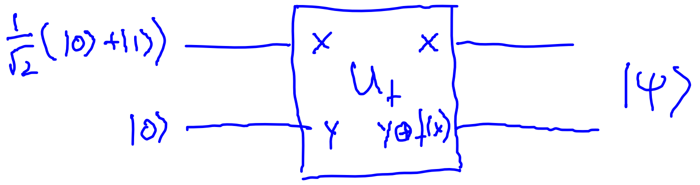
simplified expression (omit ancillas) $\boxed{(x, y) \longrightarrow (x, y \oplus f(x))}$

Quantum Function Evaluation

example: function *f* acting on one bit $f : \{0,1\} \to \{0,1\}$

quantum version of function: $U_f : |x, y\rangle \to |x, y \oplus f(x)\rangle$

circuit for evaluating *f*:



simultaneous evaluation of $f(0)$ and $f(1)$: $U_f \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|0\rangle = \underline{\frac{1}{\sqrt{2}}(|0, f(0)\rangle + |1, f(1)\rangle)}$

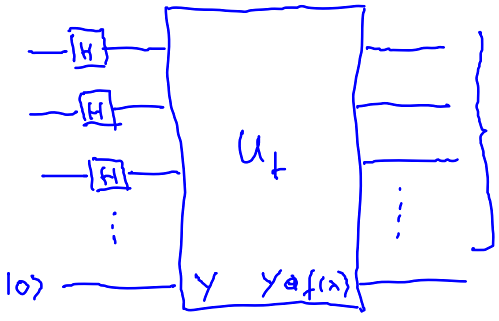
note:

- A single circuit evaluating f once does so **simultaneously** for $x = 0, 1$ making use of superposition states.
- However, upon measurement one can only extract the result of **one** of the evaluations.

generalization to many qubits:

preparation of superposition state for all input **data** qubits using Hadamard gates

preparation of **target** qubit in $|0\rangle$



$$\frac{1}{\sqrt{2^n}} \sum_x |x\rangle$$

$$\frac{1}{\sqrt{2^n}} \sum_x |x\rangle|f(x)\rangle$$

result of function evaluation

Deutsch's Problem:

evaluate if a function f is constant or balanced

$$f : \{0,1\} \longrightarrow \{0,1\}$$

x	$f_1(x)$	$f_2(x)$	$f_3(x)$	$f_4(x)$
0	0	1	0	1
1	0	1	1	0
	CONSTANT		BALANCED	

$f(0) \oplus f(1) = 0$ or 1

classically **two** queries of the function f are required to determine if it is constant or balanced.

Is there a more efficient way to solve Deutsch's problem on a quantum computer?
Yes! Make use of superposition principle and quantum function evaluation!

original work:

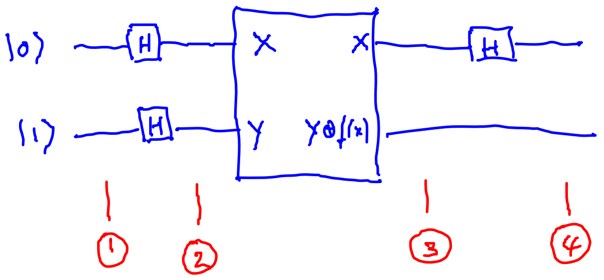
Quantum Theory, the Church-Turing Principle and the Universal Quantum Computer
D. Deutsch
Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences > Vol. 400, No. 1818 (Jul., 1985), pp. 97-117
Article Information | Page of First Match | Print | Download

Rapid Solution of Problems by Quantum Computation
David Deutsch; Richard Jozsa
Proceedings: Mathematical and Physical Sciences > Vol. 439, No. 1907 (Dec., 1992), pp. 553-558
Article Information | Page of First Match | Print | Download

Deutsch(-Josza) Algorithm

(improved version)

quantum circuit implementation:



execution of algorithm:

$$|\psi\rangle = \overset{①}{|01\rangle} \xrightarrow{H\otimes H} \frac{1}{\sqrt{2}}\left(|0\rangle+|1\rangle\right)\frac{1}{\sqrt{2}}\left(|0\rangle-|1\rangle\right) \quad ②$$

$$\xrightarrow{U_f} \begin{cases} \pm\frac{1}{2}\left(|0\rangle+|1\rangle\right)\left(|0\rangle-|1\rangle\right) & \\ & ③ \\ \pm\frac{1}{2}\left(|0\rangle-|1\rangle\right)\left(|0\rangle-|1\rangle\right) & \end{cases} \quad \begin{array}{l} \text{for } f(0)=f(1) \\ \\ \text{for } f(0)\neq f(1) \end{array}$$

$$\xrightarrow{H\otimes I} \begin{cases} \pm\frac{1}{\sqrt{2}}\,|0\rangle\left(|0\rangle-|1\rangle\right) & \\ & ④ \\ \pm\frac{1}{\sqrt{2}}\,|1\rangle\left(|0\rangle-|1\rangle\right) & \end{cases} \quad \begin{array}{l} \text{for } f(0)=f(1) \\ \\ \text{for } f(0)\neq f(1) \end{array}$$

$$= \pm\,|f(0)\oplus f(1)\rangle\,\frac{1}{\sqrt{2}}\left(|0\rangle-|1\rangle\right)$$

Measurement of first qubit reveals whether f is balanced or constant.

Notes:

- Deutsch's problem is not the most important one. It has no known (useful) applications.
- BUT it serves as a good example what a quantum computer can do.
- The Deutsch algorithm can be extended to work on an arbitrary number n of bits and determine, if a function is balanced or constant in one evaluation, whereas solving the problem deterministically takes $2^n/2 + 1$ evaluations.
- HOWEVER, on a probabilistic classical computer one could solve the problem with high probability with fewer evaluations.

Experimental Implementations in NMR and ion traps:

Chuang, I. I., Vandersypen, I. M. K., Zhou, X., Leung, D. W. & Lloyd, S.
Experimental realization of a quantum algorithm.
Nature **393**, 143-146 (1998)
|Article|

Jones, T. F. & Mosca, M.
Implementation of a quantum algorithm to solve Deutsch's problem on a nuclear magnetic resonance quantum computer.
J. Chem. Phys. **109**, 1648-1653 (1998)
|Article|

Gulde S, Riebe M, Lancaster GPT, et al.
Implementation of the **Deutsch**-Jozsa **algorithm** on an ion-trap quantum computer
NATURE 421 (6918): 48-50 JAN 2 2003

Public Key Cryptosystems:

task: Alice wants to receive a message from Bob (or anybody else, for that matter) and keep it secret. She supplies a **public key** (P) to everybody that wants to send her messages. The public key is used to **encrypt** the message to be sent using a scheme that is very difficult to reverse. After encryption of the message **only** Alice can efficiently **decrypt** it using her **secret private key** (S).

note: Public crypto systems such as RSA are only **believed but not proven to be secure**, even though a lot of effort has gone into examining the question of security of RSA.

RSA (Rivest, Shamir, Adleman) Cryptosystem:

An equivalent cryptosystem was developed by the UK intelligence agency GCHQ in 1973. This fact was only revealed in 1997, as it was classified.

A method for obtaining digital signatures and public-key cryptosystems
R. L. Rivest, A. Shamir, L. Adleman
Communications of the ACM archive, Volume 21 , Issue 2 (February 1978) Pages: 120 - 126
Full text available: Pdf (749KB) full citation, abstract, references, citings, index terms

protocol for generating the public and secret keys:

- select to prime numbers p and q
- compute the product $n = p\,q$
- select a small integer e that is relatively prime to $\varphi(n) = (p-1)(q-1)$
- compute d, the multiplicative inverse of $e \bmod \varphi(n)$
- the RSA public key is the pair $P = (e,n)$ and the secret key is the pair $S=(d,n)$

notes:

- prime numbers p and q can be found efficiently by guessing and testing if the number is prime
- the probability of a number with L bits to be prime is roughly $1/L$ (that is large)
- to test if it is prime using a primality test requires about $O(L^3)$ operations

Thus, key generation is efficient with $O(L^4)$ operations required.

protocol for encoding and decoding a message:

- assume that the message M has $\log n$ bits
- the encrypted message E is calculated as $E(M) = M^e \bmod n$
- to decode the message $D(E(M)) = E(M)^d \bmod n$ is calculated

note: modular exponentiation can be done efficiently with $O(L^3)$ operation as well

for examples see:

- http://en.wikipedia.org/wiki/RSA
- Weisstein, Eric W. "RSA Encryption." From *MathWorld*--A Wolfram Web Resource. http://mathworld.wolfram.com/RSAEncryption.html

How to break RSA?

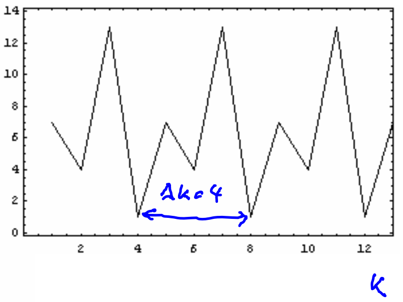
- find the prime factors of n
- use a quantum computer

An algorithm for factoring 15 classically (in a complicated way)

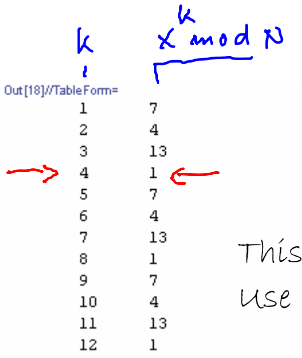
the algorithm:

step 0: choose a number to be factored, here $N=15$
step 1: chose at random a number x that has no common factors with N
step 2: compute the order r of x with respect to N

definition of order: $x^r = 1 \pmod N$



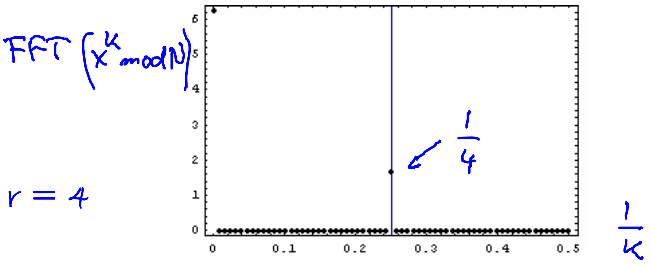
values:



```
Out[18]//TableForm=
 1    7
 2    4
 3    13
 4    1
 5    7
 6    4
 7    13
 8    1
 9    7
10    4
11    13
12    1
```

This is a periodic function in k with period $r = 4$.
Use Fourier transform to find period r.

fast Fourier transform (FFT):
requires $O(n2^n)$ classical gates to calculate Fourier transform of 2^n complex numbers



order is $r = 4$

---

the order r of $x = 7$ mod N with $N = 15$ is $r = 4$ (an even number):

step 4: compute

$$x^{r/2} \bmod N = 7^2 \bmod 15 = 4 \neq -1 \bmod 15$$

Algorithm has succeeded! Otherwise choose different x and start again!

step 5: calculate greatest common devisor (gcd) of $x^2 + 1$ and $x^2 - 1$ with N:

$$\gcd(x^2+1, N) = \gcd(50, 15) = \gcd(2 \cdot 5^2, 3 \cdot 5) = \underline{\underline{5}}$$

$$\gcd(x^2-1, N) = \gcd(48, 15) = \gcd(2^4 \cdot 3, 3 \cdot 5) = \underline{\underline{3}}$$

result: $\boxed{15 = 3 \cdot 5 \quad !}$

This algorithm provides an **exponential speed up** in comparison to any other known classical algorithm to find the prime factors of a number N when the Fourier transform is implemented as a **quantum Fourier transform** on a quantum computer.

Definition of the discrete Fourier transform (DFT):

input: vector of complex numbers with length N and elements $x_0, x_1, \dots, x_{N-1}$

output: vector of complex numbers with length N and elements $y_0, y_1, \dots, y_{N-1}$
with

$$\underline{y_k} = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} x_j \, e^{2\pi i \, jk/N}$$

Definition of the **quantum Fourier transform** (QFT):
a linear operator acting on the basis states of an orthonormal basis $|j\rangle = |0\rangle, \dots, |N-1\rangle$
as defined by

$$|j\rangle \longrightarrow \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{2\pi i \, jk/N} |k\rangle$$

equivalently the action of the QFT on an arbitrary state is given by:

$$|\psi\rangle = \sum_{j=0}^{N-1} x_j |j\rangle \longrightarrow \sum_{k=0}^{N-1} \underline{y_k} |k\rangle$$

Product representation of the QFT:

Consider a quantum computer with n bits with the computational basis $|0\rangle, \dots, |2^n-1\rangle$ and $N = 2^n$.

for example $n = 3$:

$$|j\rangle : \quad \begin{array}{lcl} |0\rangle & \equiv & |000\rangle \\ |1\rangle & & |001\rangle \\ |2\rangle & & |010\rangle \\ |3\rangle & & |011\rangle \\ |4\rangle & & |100\rangle \\ |5\rangle & & |101\rangle \\ |6\rangle & & |110\rangle \\ |7\rangle & & \underline{|111\rangle} \end{array}$$

$$N = 2^3 = 8$$

Let's use a binary representation for the basis state $|j\rangle$.

$$|j\rangle : \quad j = j_1 j_2 \dots j_n$$

$$j = j_1 2^{n-1} + j_2 2^{n-2} + \dots + j_n 2^0$$

and the **binary fraction** is defined as:

$$0.j_l\, j_{l+1} \ldots j_m = \frac{j_l}{2} + \frac{j_{l+1}}{4} + \ldots + \frac{j_m}{2^{m-l+1}}$$

for example:

$0.j_3$:
$0.0 = 0$
$0.1 = \frac{1}{2}$

$0.j_2 j_3$:
$0.00 = 0$
$0.01 = 1/4$
$0.10 = 2/4$
$0.11 = 3/4$

$0.j_1 j_2 j_3$:
$0.000 = 0$
$0.001 = 1/8$
$0.010 = 2/8$
$0.011 = 3/8$
$0.100 = 4/8$
$0.101 = 5/8$
$0.110 = 6/8$
$0.111 = 7/8$

then the QFT can be written down in the following product form:

$$|j_1, \ldots, j_n\rangle \longrightarrow \frac{1}{\sqrt{2^n}} \Big[\left(|0\rangle + e^{2\pi i\, 0.j_n} |1\rangle \right) \left(|0\rangle + e^{2\pi i\, 0.j_{n-1} j_n} |1\rangle \right) \ldots \left(|0\rangle + e^{2\pi i\, 0.j_1 j_2 \cdots j_n} |1\rangle \right) \Big]$$

for proof see Nielsen & Chuang p. 218

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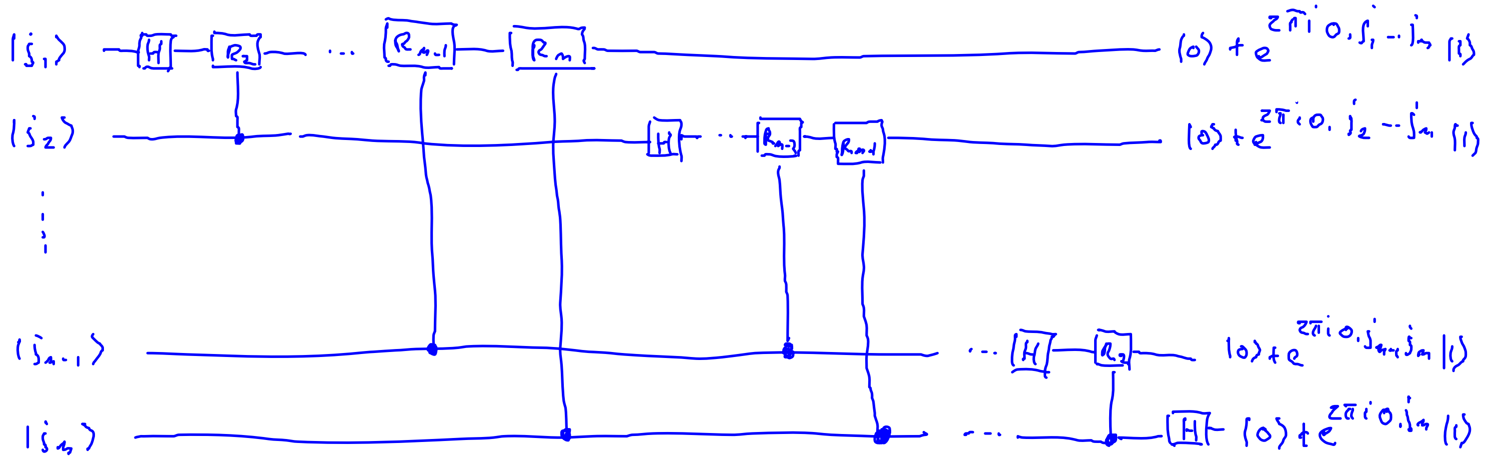
circuit representation of quantum Fourier transform:



with the Hadamard transform H and the unitary operation

$$R_k = \begin{pmatrix} 1 & 0 \\ 0 & e^{2\pi i/2^k} \end{pmatrix}$$

How does this circuit work?

apply Hadamard to first qubit:

$$\frac{1}{\sqrt{2}} \left(|0\rangle + \underbrace{e^{2\pi i\, 0.j_1}} |1\rangle \right) |j_2 \ldots j_n\rangle$$

$0.0 : e^{2\pi i\, 0.0} = e^0 = 1$
$0.1 , e^{2\pi i \frac{1}{2}} = e^{\pi i} = -1$

apply controlled R_2 to first qubit:

$$\frac{1}{\sqrt{2}} \left(|0\rangle + e^{2\pi i \, 0.j_1 j_2} |1\rangle \right) |j_2 \dots j_n\rangle$$

apply R_3 to R_n the first qubit:

$$\frac{1}{\sqrt{2}} \left(|0\rangle + e^{2\pi i \, 0.j_1 j_2 \dots j_n} |1\rangle \right) |j_2 \dots j_n\rangle$$

Done with first qubit move on to second one. Apply H and then R_3 to R_{n-1}

$$\xrightarrow{H_2} \quad \frac{1}{\sqrt{2^2}} \left(|0\rangle + e^{2\pi i \, 0.j_1 j_2 \dots j_n} |1\rangle \right) \left(|0\rangle + e^{2\pi i \, 0.j_2} |1\rangle \right) |j_3 \dots j_n\rangle$$

$$\xrightarrow{R_3 \dots R_{n-1}} \quad \frac{1}{\sqrt{2^2}} \underbrace{\left(|0\rangle + e^{2\pi i \, 0.j_1 j_2 \dots j_n} |1\rangle \right)}_{\text{qubit 1}} \underbrace{\left(|0\rangle + e^{2\pi i \, 0.j_2 \dots j_n} |1\rangle \right)}_{\text{qubit 2}} |j_3 \dots j_n\rangle$$

Continue until done. Then apply **SWAP** gates to invert order of qubits to get the definition of the **QFT** on the previous page.

How many gates does this circuit use for calculating the Fourier transform?

- 1 Hadamard gate for each of the n qubits
- $n - 1$ conditional rotations for the first, $n - 2$ for the second ...,

total:

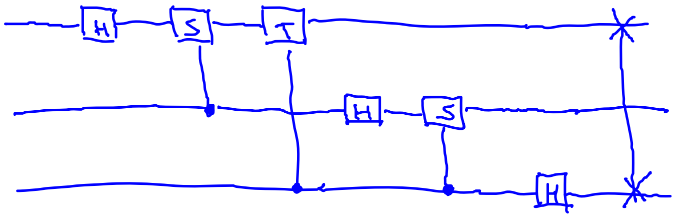
$$n + (n-1) + (n-2) + \dots + 1 = n(n+1)/2 = O(n^2)$$

In addition $n/2$ SWAP gates (composed of 3 CNOT gates) are used. But this does not change the scaling of the number of gates required for calculating the QFT.

Thus the **quantum Fourier transform** (QFT) requiring $O(n^2)$ gates provides and **exponential speed up** over the fast Fourier transform (FFT) requiring $O(n2^n)$ gates for calculating the Fourier transform of a n-bit complex number.

That is a great result, BUT the information stored in the complex probability amplitudes cannot be directly accessed by single measurements. More subtle approaches are needed.

Example: 3 qubit quantum Fourier transform



$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \; ; \; S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \; ; \; T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$$

$$= R_2 \qquad\qquad = R_3$$

exercise: Work out the matrix representation for this circuit in the computational basis.

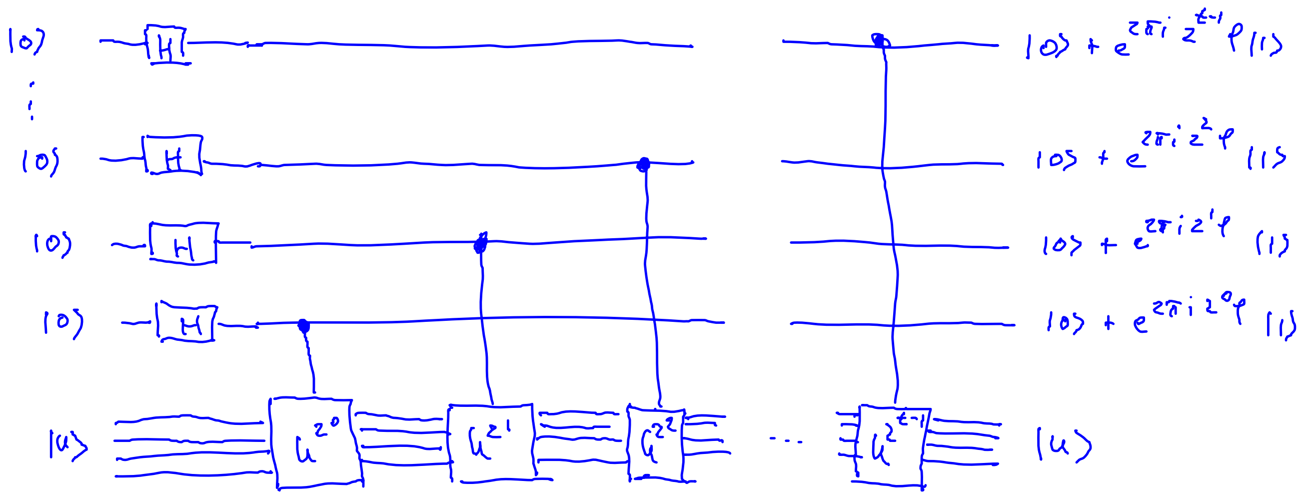
Quantum Phase Estimation

a subroutine for making use of the QFT, e.g. in the factoring algorithm

goal: find phase φ of eigenvalue u of an operator U with $U|u\rangle = u|u\rangle = e^{2\pi i\varphi}|u\rangle$

assumption: some black box routine exists which realizes the operator U²^j for some integer j > 0 and an eigenstate |u> of U can be prepared.

quantum circuit for first stage of phase estimation:



operation of phase estimation:

- apply Hadamard gate to all bits in first register
- do controlled U^{2^n} operations on all bits

final state of 1st register:

$$\frac{1}{\sqrt{2^t}} \left(|0\rangle + e^{2\pi i 2^{t-1}\varphi}|1\rangle\right)\left(|0\rangle + e^{2\pi i 2^{t-2}\varphi}|1\rangle\right) \ldots \left(|0\rangle + e^{2\pi i 2^{0}\varphi}|1\rangle\right)$$

$$= \frac{1}{\sqrt{2^t}} \sum_{k=0}^{2^t-1} e^{2\pi i \varphi k}|k\rangle$$

2nd register remains unchanged: $|u\rangle$

assume φ to be represented as t bit binary number:

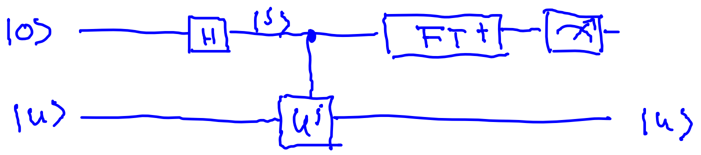
$$\frac{1}{\sqrt{2^t}} \left(|0\rangle + e^{2\pi i 0.\varphi_t}|1\rangle\right)\left(|0\rangle + e^{2\pi i 0.\varphi_{t-1}\varphi_t}\right) \ldots \left(|0\rangle + e^{2\pi i 0.\varphi_1 \ldots \varphi_t}|1\rangle\right)$$

This expression has the form of a QFT. Apply inverse QFT and measure all bits in first register:

$$\overset{!}{=} |\varphi_1 \varphi_2 \ldots \varphi_t\rangle = |\varphi\rangle$$

---

circuit representation of phase estimation:



But still need to find U and $|u\rangle$ for the specific application, e.g. factoring!

Quantum Order Finding:

- for integers x, N with no common factors and ($x<N$) find smallest r such that:

$$x^r = 1 \pmod N$$

- no classical algorithm is believed to exists that can find r in time polynomial in $L=\log(N)$
- application of the phase estimation algorithm to the unitary operator

$$U|y\rangle = |x\,y \bmod N\rangle \qquad \text{with } y \in \{0,1\}^L$$

(i.e. multiply y with x and take it mod N) can solve the problem efficiently on a quantum computer

choose eigen state of U as:

$$|u_s\rangle = \frac{1}{\sqrt{r}}\sum_{k=0}^{r-1}\exp\left[-\frac{2\pi i s k}{r}\right]|x^k \bmod N\rangle \quad \text{for} \quad 0 \le s \le r-1$$

with:

$$U|u_s\rangle = \frac{1}{\sqrt{r}}\sum_{k=0}^{r-1}\exp\left[-\frac{2\pi i s k}{r}\right]|x^{k+1} \bmod N\rangle$$

$$= \underline{\exp\left[\frac{2\pi i s k}{r}\right]}\,|u_s\rangle$$

eigenvalue can be determined accurately with phase estimation algorithm

---

Efficiency of Modular Exponentiation

efficient implementation of controlled U^{2^j} operations on t bit representation of z is required:

$$|z\rangle|y\rangle \longrightarrow |z\rangle\, U^{z_t 2^{t-1}} \ldots U^{z_1 2^0}|y\rangle$$

$$= |z\rangle\,|x^{z_t 2^{t-1}} \cdots x^{z_1 2^0}\, y \bmod N\rangle$$

$$= |z\rangle\,|x^z\, y \bmod N\rangle$$

i.e. equivalent to multiplying second register y with $x^z \bmod N$

implementation of modular exponentiation of x (first step):

$$x^2 \bmod N \longrightarrow (x^2 \bmod N)^2 \bmod N \longrightarrow \ldots \quad \text{up to} \quad x^{2^j} \bmod N \quad \text{for } j = t-1$$

at cost: $\mathcal{O}(L)$ squaring operations for $\mathcal{O}(L^2)$ operations each

second step:

$$x^z \bmod N = (x^{z_t 2^{t-1}} \bmod N)(x^{z_{t-1} 2^{t-2}} \bmod N)\ldots(x^{z_1 2^0} \bmod N)$$

at cost: $\mathcal{O}(L^3)$

initial state $|u_s>$

- preparing the initial state $|u_s>$ would require knowledge of r which we seek to determine in the order finding
- but we observer that

$$\frac{1}{\sqrt{r}} \sum_{s=0}^{r-1} |u_s\rangle = |1\rangle$$

- this is a superposition of eigenstates of U and thus it is also an eigenstate which we can use for the phase estimate algorithm
- it is trivial to prepare the initial state $|1>$
- in the phase estimation algorithm for each s in the range $[0, \dots ,r-1]$ we obtain an estimate of the phase $\varphi \approx s/r$
- the estimate of φ is accurate to $2L + 1$ bits
- find r using the continued fraction algorithm

The continued fractions algorithm

a real number expressed in terms of integer fractions

$$[a_0, \dots, a_M] \equiv a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\dots + \cfrac{1}{a_M}}}}$$

m^{th} convergent $(0 \leq m \leq M)$:

$$[a_0, \dots, a_m]$$

example:

$$\frac{31}{13} = 2 + \frac{5}{13} = 2 + \cfrac{1}{\frac{13}{5}} = 2 + \cfrac{1}{2 + \frac{3}{5}} = 2 + \cfrac{1}{2 + \cfrac{1}{\frac{5}{3}}}$$

$$= 2 + \cfrac{1}{2 + \cfrac{1}{1 + \frac{2}{3}}} = 2 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{3/2}}} = 2 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \frac{1}{2}}}}$$

How many operations to determine the continued fraction representation of a real number?

for $\varphi = \frac{s}{r}$ with s, r : integers with L bits then $O(L^3)$ operations are required

Algorithm

inputs:

- $U_{x,N}: \ |j\rangle|k\rangle \longrightarrow |j\rangle|x^j k \bmod N\rangle$ black box

  for x co-prime to the L bit number N

- $t = 2L+1+\lceil \log(2+\frac{1}{2\varepsilon})\rceil$ qubits initialize in $|0\rangle$
- L qubits initialized in state $|1\rangle$

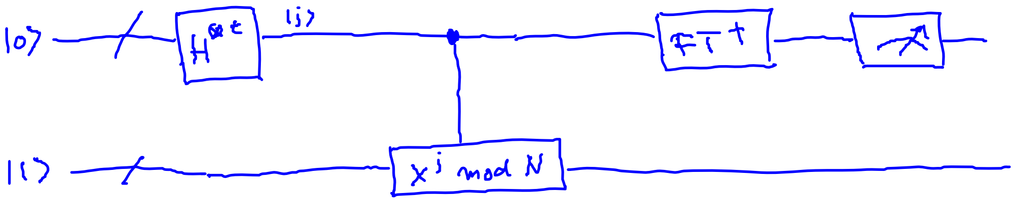
1. initial state $|0\rangle|1\rangle$
2. create superposition $\longrightarrow \frac{1}{\sqrt{2^t}}\sum_{j=0}^{2^t-1}|j\rangle|1\rangle$
3. apply $U_{x,N}$ $\longrightarrow \frac{1}{\sqrt{2^t}}\sum_{j=0}^{2^t-1}|j\rangle|x^j \bmod N\rangle = \frac{1}{\sqrt{r2^t}}\sum_{s=0}^{r-1}\sum_{j=0}^{2^t-1}e^{2\pi i s\frac{j}{r}}|j\rangle|u_s\rangle$
4. apply inverse QFT $\longrightarrow \frac{1}{\sqrt{r}}\sum_{s=0}^{r-1}|\widetilde{\frac{s}{r}}\rangle|u_s\rangle$
5. measure first register $\longrightarrow |\widetilde{\frac{s}{r}}\rangle$
6. apply continued fractions algorithm $\longrightarrow |r\rangle$

---

quantum circuit for order finding

register 1
t qubits

register 2
L qubits



Quantum Factoring Algorithm

input: a composite number N
output: a non-trivial factor of N
runtime: $O((\log N)^3)$

1. if N even, return factor 2
2. determine whether $N = a^b$ for $a \geq 1$ and $b \geq 2$, if yes return a
3. choose random integer x in interval [1 ... N-1]. Return gcd(x,N), if gcd(x,N) > 1
4. find order r of x mod N using the quantum order finding algorithm
5. if r is even (will be the case with probability 1/2) and $x^{r/2} \neq -1 \mod N$ then compute $\gcd(x^{r/2}-1,N)$ and $\gcd(x^{r/2}+1,N)$ and test if one of these Is non-trivial factor of N. Otherwise choose new x and repeat.

Factoring 15 using Quantum Order Finding

input state: $|0\rangle|0\rangle$

apply Hadamard transform to first register:

$$\frac{1}{\sqrt{2^t}} \sum_{k=0}^{2^t-1} |k\rangle|0\rangle = \frac{1}{\sqrt{2^t}} \left[|0\rangle + |1\rangle + |2\rangle + \dots + |2^t-1\rangle \right] |0\rangle$$

choose x = 7 (as before) and calculate xk mod N and leave result in second register:

$$\frac{1}{\sqrt{2^t}} \sum_{k=0}^{2^t-1} |k\rangle |x^k \bmod N\rangle = \frac{1}{\sqrt{2^t}} \Big[|0\rangle|1\rangle + |1\rangle|7\rangle + |2\rangle|4\rangle + |3\rangle|13\rangle + |4\rangle|1\rangle + |5\rangle|7\rangle + |6\rangle|4\rangle + \dots \Big]$$

Measure second register. One of the states $|1>$, $|7>$, $|4>$, $|13>$ will be found. Suppose we would have found $|4>$ with probability 1/4. Thus the state at the input of the FFT would have been:

$$\sqrt{\frac{4}{2^t}} \left[|2\rangle + |6\rangle + |10\rangle + |14\rangle + \dots \right] \xrightarrow{FT^\dagger} \sum_\ell \alpha_\ell |\ell\rangle$$

Quantum Search Algorithms (Grover's Algorithm)

search problems:

Traveling salesman problem: Find the shortest route between that passes through all of a set of given cities. If there is N possible routes a classical computer will need $O(N)$ steps to find the shortest route, by evaluating all lengths and keeping one record of the shortest one.

Using a **quantum search algorithm** this problem can be solved in $O(N^{1/2})$ steps. Note that it is not capable of providing an exponential speed up. It can be shown that the $N^{1/2}$ is the best efficiency that can be reached by any quantum algorithm.

Searching an unstructured data base: Finding an entry with certainty in an unstructured database with N elements classically takes N queries of the database. With quantum search algorithm it takes $N^{1/2}$ queries.

structured data base: It is easy to find an entry in a structured (ordered) data base such as a telephone book that is sorted by names. If the phone book has $N=2^n$ entries it takes $n = \log N$ steps to find the desired number.

unstructured data base: To find the name corresponding to a certain number will take N queries to succeed with probability $P = 1$ or 2^{n-1} queries with $P = 1/2$.

Oracles:

definition of search problem: Find a single entry in a data base having $N = 2^n$ entries. Assign a unique index x in the range $1, \ldots, N$ that can be stored in n bits to each item of the database. Define a function f with $f(x) = 1$ if x is a solution to the search problem and $f(x) = 0$ otherwise.

definition of a quantum oracle:
A (quantum) black box that can recognize solutions to the search problem using an oracle qubit $|q>$. The oracle is realized as a unitary operator O defined as

$$|x\rangle|q\rangle \xrightarrow{O} |x\rangle\,|q \oplus f(x)\rangle$$

If x is a solution to the search problem the oracle will flip the value of the oracle qubit.

Prepare the oracle qubit in a superposition state

$$|x\rangle \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right) \longrightarrow |x\rangle \frac{1}{\sqrt{2}}\left(|0 \oplus f(x)\rangle - |1 \oplus f(x)\rangle\right)$$

$$\longrightarrow \begin{array}{ll} \frac{1}{\sqrt{2}}\,|x\rangle\left(|0\rangle - |1\rangle\right) & \text{for } f(x) = 0 \\ \frac{1}{\sqrt{2}}\,\underline{\underline{-}}|x\rangle\left(|0\rangle - |1\rangle\right) & \text{for } f(x) = 1 \end{array}$$

generally:

$$|x\rangle \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \xrightarrow{O} (-1)^{f(x)} |x\rangle \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

The state acquires a **phase shift** thus the oracle **marks** the state to be found. The oracle qubit stays in same state thus it can be omitted from the discussion.

$$|x\rangle \xrightarrow{O} (-1)^{f(x)} |x\rangle$$

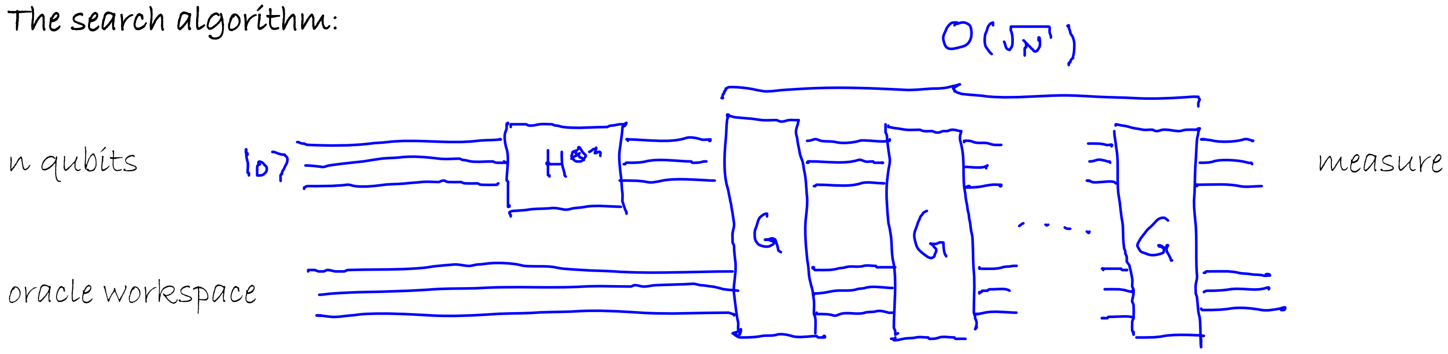
note: The oracle can **recognizes** the state to be found but of course does **not know** it beforehand.

e.g. factoring: Search for number p that is a prime factor of m by starting at $x = 1$ and dividing m by x for x up to $m^{1/2}$. A prime factor can be recognized easily but it is hard to find.

oracle construction: Use ideas of classical reversible computation to construct oracle function that results in $f(x) = 1$ when finding the result and $f(x) = 0$ otherwise and implement it quantum mechanically.

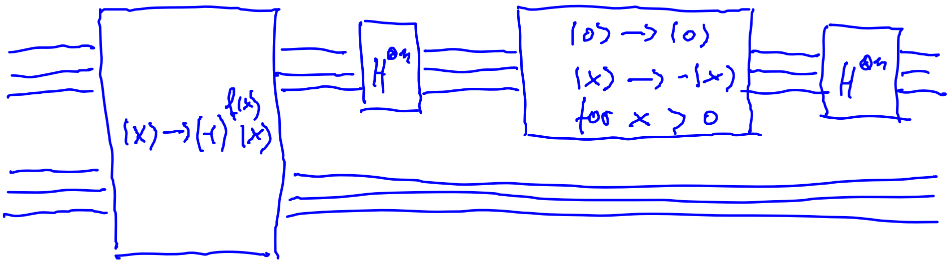
$$(x, q) \longrightarrow (x, q \oplus f(x))$$

The search algorithm:



- initial state $|0\rangle^{\otimes n}$
- prepare input states in superposition $\frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} |x\rangle$
- apply the **Grover iteration** repeatedly

Grover iteration:



procedure:

- apply the oracle switching the phase of the input register for $f(x) = 1$
- apply the Hadamard transform on the input register
- apply a conditional phase shift to every input basis state except 0

$$|x\rangle \longrightarrow -(-1)^{\delta_{x0}}|x\rangle$$

- apply the Hadamard transform on the input register

efficiency:

- both Hadamard transforms on the input register require $n = \log N$ gates
- the conditional phase shift can be implemented using $O(n)$ gates, c.f. CNOT
- implementation of the oracle is classical, its efficiency depends on the task of the oracle.

(Frequently it is easy to test if a result is a solution to the problem, but it is hard to find the solution as in factoring for example.)

Generalized form of the Grover Iteration

combined action of steps 2 - 3:

$$H^{\otimes n}(2|0\rangle\langle 0| - I)H^{\otimes n} = 2|\psi\rangle\langle\psi| - I$$

where $$|\psi\rangle = \frac{1}{\sqrt{N}}\sum_{x=0}^{N-1}|x\rangle$$

thus $$G = (2|\psi\rangle\langle\psi| - I)\overset{\text{oracle}}{O}$$

remember:

- H is self adjoint

$$H^\dagger = H \; ; \quad H^\dagger H = I = HH$$

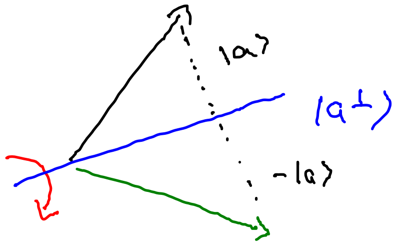
Geometric interpretation

assume $x = a$ is a solution to the search algorithm such that $f(a) = 1$.

action of the oracle:

$$|x\rangle \to |x\rangle$$
$$|a\rangle \to -|a\rangle$$

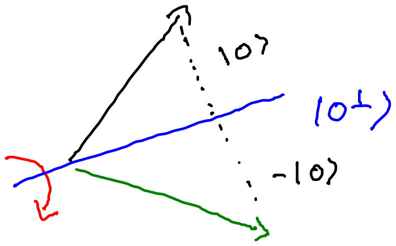
looks like a reflection about a (hyper) plane orthogonal to $|a>$



action of the conditional phase shift: $-(I - 2|0\rangle\langle 0|)$

$$|x\rangle \to |x\rangle$$
$$|0\rangle \to -|0\rangle$$

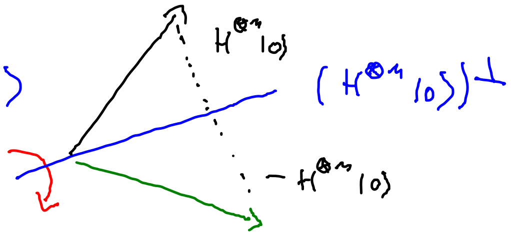
looks like a reflection about a (hyper) plane orthogonal to $|0>$



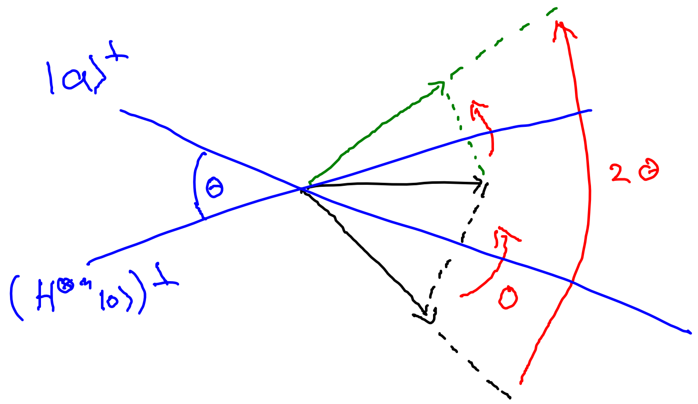
Hadamard gates:

$$-H^{\otimes n}(I - 2|0\rangle\langle 0|)H^{\otimes n} = -(I - 2H^{\otimes n}|0\rangle\langle 0|H^{\otimes n})$$

i.e. this is a reflection to a hyper plane orthogonal to $H^{\otimes n}|0\rangle$



The Grover iteration corresponds to two concatenated reflections, i.e. a rotation.

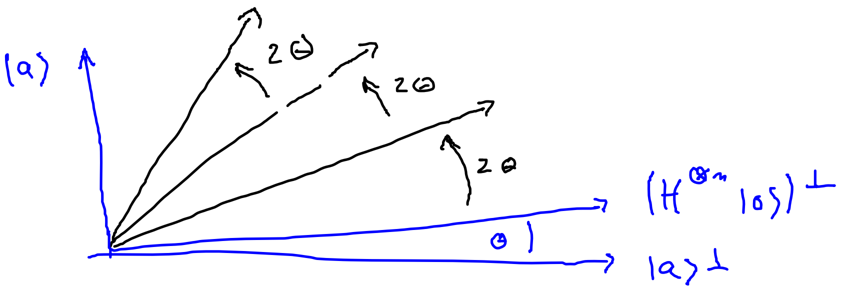


i.e. the state vector of the system gets rotated from its initial state towards the searched state $|a>$ in every iteration of the search algorithm.

angle θ:

$$|\langle a|H^{\otimes n}|0\rangle| = \frac{1}{\sqrt{2^n}} = \sin\theta$$

number of iterations r required to find result:



- rotation angle after r iterations: $\Theta + 2r\Theta$

for large n

- individual rotation angle: $|\langle a|H|0\rangle| = \frac{1}{\sqrt{2^n}} = \sin\Theta \approx \Theta$

- iterate until $(2r+1)\Theta \approx \frac{\pi}{2}$

$$\Rightarrow r \approx \frac{\pi}{4}\sqrt{2^n} = \underline{\underline{\mathcal{O}(\sqrt{N})}}$$

The Grover quantum search algorithm provides a **quadratic** speedup in comparison to classical search algorithms.

---

example: To break data encryption with a **56 bit** key in the DES (Data Encryption Standard) scheme by searching for the key classically would require to try $2^{56} = 7\,10^{16}$ keys. Checking keys at a rate of 1 million per second would require a classical computer more than **a thousand years**. On a quantum computer that can check the same number of keys per second it would only take **4 minutes**.

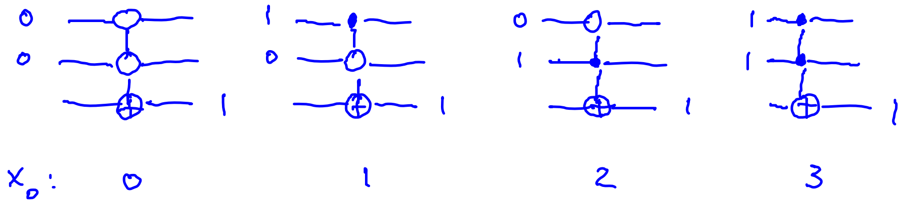
(Of course we could chose to encrypt our data using a 112 bit key.)

note:

- the scheme discussed above can be extended to cases when there is more than one solution to the search problem
- the scheme can be generically applied to a large variety of search problems by constructing an oracle for the problem and executing the Grover algortihm.

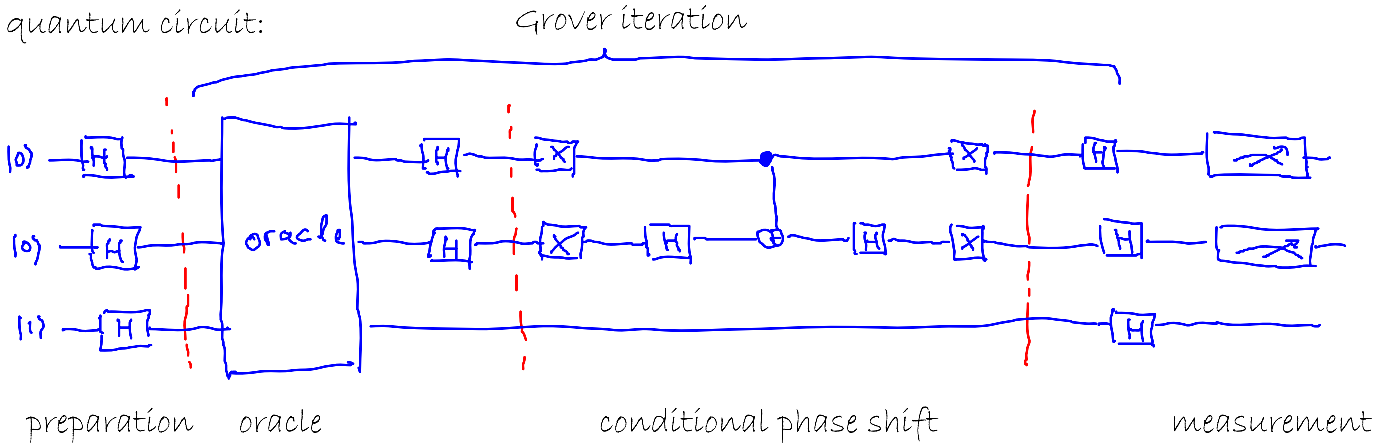
Quantum Search Example

search space $N = 2^2 = 4$.



task: Distinguish between the four oracles with only one query, i.e. find the $x = x_0$ for which the oracle is true.

quantum circuit:



exercise: Show that x_0 can be found with only one query of the oracle !

Search in an unstructured database

task: Assume a database with $N = 2^n$ entries of length l bits each. Each entry is labeled with $d_1, \ldots, d_N$. Find the label of the l bit string that matches the string s to be found.

approach: A **central processing unit** (CPU) performs the operations on a small amount of temporary memory. This memory is usually to small to store the whole data base. Therefore, the (large) database of size $N\,l$ is stored in the **memory** part of the computer. As a result data needs to be **loaded** from the memory into the processor and data from the processor needs to be **stored** in the memory.

classical solution:

- set up an index with $n = \log N$ bits for the N elements of the database in the CPU memory
- load first entry from database
- compare to string s
- increment index by 1
- halt when string is found

In the worst case 2^n queries to the database need to be performed.

ingredients for quantum approach:

processor:

- n qubit register $|x\rangle$ initialized to $|0\rangle$ to store the index to the data base
- l qubit register initialized to $|s\rangle$ and remaining in that state
- l qubit data register $|d\rangle$ initialized to $|0\rangle$
- 1 qubit register initialized to $(|0\rangle - |1\rangle)/2^{1/2}$

quantum memory:

- N cells with l qubits each storing the states $|d_x\rangle$

BUT quantum memory is not so stable. Decoherence can destroy the state of the memory.
What about using classical memory instead?

classical memory:

- N cells with l bits each storing the states $|d_x\rangle$
- as an extra feature the classical memory needs to be addressed in superpositions of register indices x
- this way superpositions of cell values can be loaded to the CPU

loading data base register $|d_x\rangle$ with index x to data register $|d\rangle$:

$$|d\rangle \longrightarrow |d \oplus d_x\rangle$$

---

implementation of oracle:

- initial state: $|x\rangle|s\rangle|0\rangle \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$
- load $|x\rangle|s\rangle|d_x\rangle \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$
- comparison of second and third register:

$$|x\rangle|s\rangle|x_d\rangle \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = \begin{cases} -|x\rangle|s\rangle|d_x\rangle \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) & \text{for } d_x = s \\ |x\rangle|s\rangle|d_x\rangle \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) & \text{for } d_x \neq s \end{cases}$$

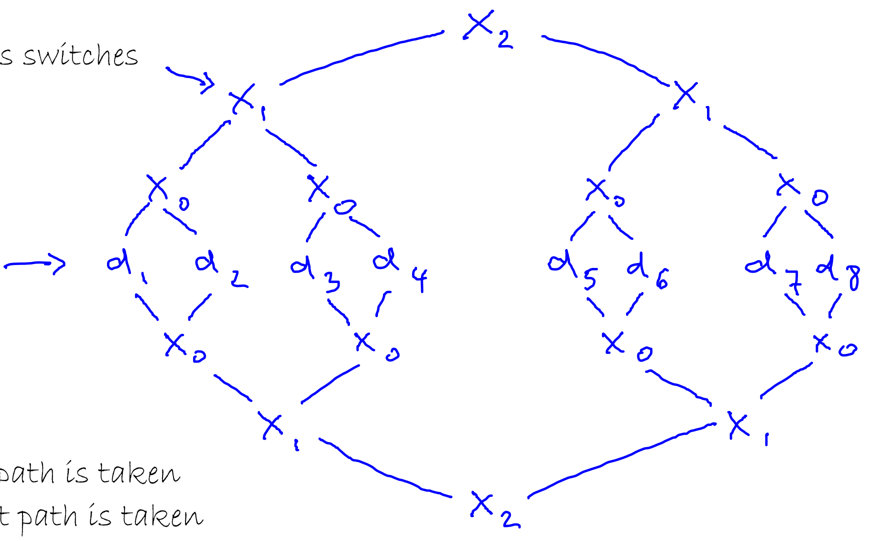
- load again $\pm |x\rangle|s\rangle|0\rangle \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

result:

- last three registers remain unentangled with $|x\rangle$
- $|x\rangle$ changes sign when $d_x = s$
- i.e. a good oracle has been found

Use Grover's algorithm to find the index $|x\rangle$ for which $d_x = s$

quantum addressing scheme for classical memory:



operation:

- if x_i is in state $|0>$ the left path is taken
- if x_i is in state $|1>$ the right path is taken
- if x_i is in state $(|0> + |1>)/2^{1/2}$ then an equal superposition of both paths is taken
- data qubits are routed according to the tree to the classical memory where the qubit state is flipped if the classical memory bit is 1 and does nothing otherwise
- then the tree is inverted moving the qubits back to definite positions leaving them with the retrieved information

possible implementation: Photons for the data register and non-linear controlled beam splitters for the tree. If memory is in state 0 photon state remains unchanged, otherwise polarization is changed by 90 degrees.

---

notes:

- Many databases are not ordinarily unstructured (phone books).
- When sorted the entries can be found in $O(\log N)$ steps.
- For unstructured data bases Grover's algorithm might help.
- For addressing a classical database quantum mechanically $N \log N$ quantum switches are required. This is about as many switches as bits required to implement the data base. Will only be useful, if making quantum switches is easy and cheap.

conclusion:

Currently it seems that the main use of quantum searching might be to find solutions to hard problems that can be mapped to a search problem by designing an appropriate oracle (e.g. as the traveling salesman problem).

Open and closed quantum systems

closed quantum systems:
- systems that **do not interact** in undesirable ways with the outside world (environment)

environment:
- a description of the outside world

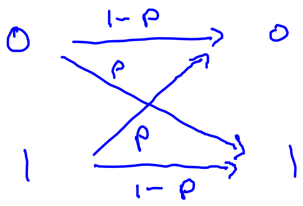
note:
- any closed quantum system must be fully decouple from the environment
- o.k. assumption for thinking about quantum computation in principle
- **BUT** any realistic physical quantum system does interact with the outside world
- such interactions are also needed for control and readout

open quantum systems:
- systems that do interact with the environment
- undesired interactions show up as noise in quantum information processing

---

Classical Noise

example: a bit stored in a memory (e.g. hard disk)



- p is probability for the bit to flip because of some noise process
- 1-p is the probability for the bit to stay in the same state

In the example of a hard disk the bit flip may be triggered by a fluctuating magnetic field in the environment of the bit.

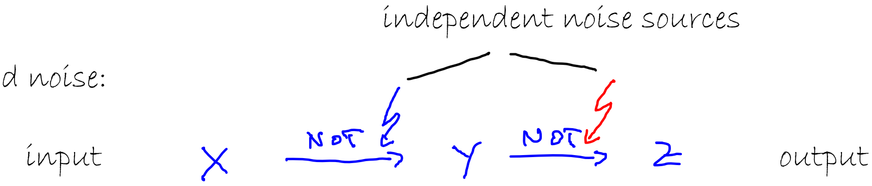
A **model of the environment and its interaction with the bit** needs to be found to understand the process. In the case of the hard disk this would consist of determining the properties of the fluctuating field in the environment and determining its interaction with the bit (using Maxwell equations).

- initial probabilities p_0 and p_1 of bit states
- final probabilities q_0 and q_1 of bits after noise has occurred
- p is the transition probability

$$\begin{pmatrix} q_0 \\ q_1 \end{pmatrix} = \begin{pmatrix} 1-p & p \\ p & 1-p \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \end{pmatrix} = \begin{pmatrix} (1-p)\, p_0 + p\, p_1 \\ p\, p_0 + (1-p)\, p_1 \end{pmatrix}$$

Correlations of Noise Processes

uncorrelated noise:



Uncorrelated stochastic noise process (X ⟶ Y ⟶ Z). This is also called a **Markov process**. The noise acts independently with no spatial or temporal correlations.

more general:

$$\vec{q} = E\,\vec{p}$$

output probabilities

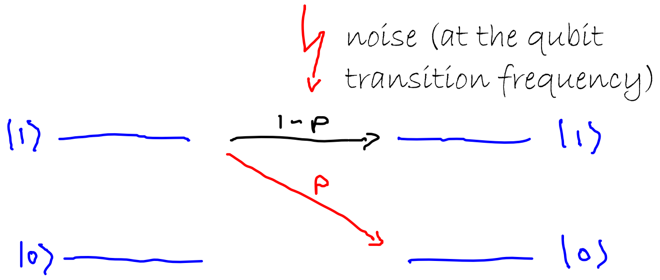
matrix of transition probabilities: evolution matrix

input probabilities

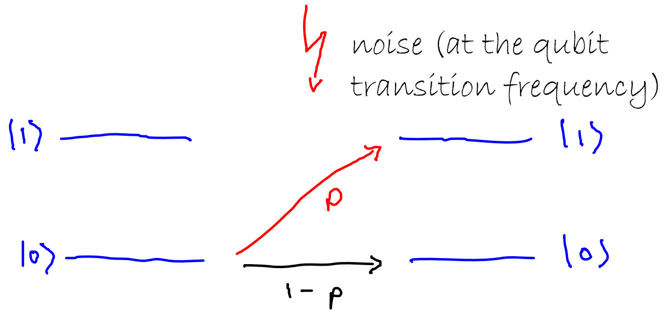
properties of E:
- linear
- non-negative entries (positive probabilities)
- completeness (column entries sum to one, total probability is conserved)

Noise Processes Acting on Qubits

relaxation



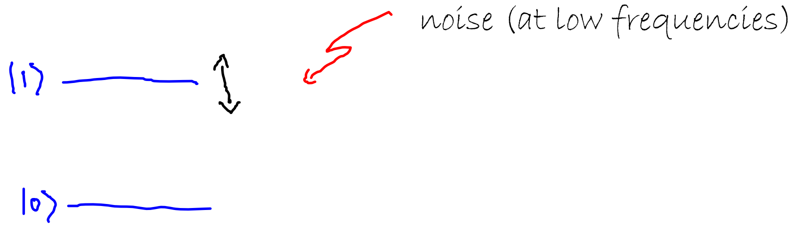
excitation



$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle \longrightarrow |\psi\rangle' = \alpha'\,|0\rangle + \beta'\,|1\rangle$$

relaxation and excitation by noise processes change the qubit ground state $|0\rangle$ and excited state $|1\rangle$ occupation probabilities $|\alpha|^2$ and $|\beta|^2$

Dephasing



Changes relative phase **φ** between ground and excited state without changing occupation probabilities.

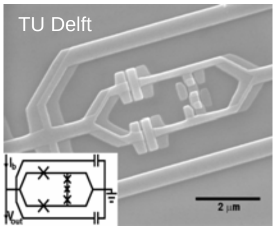
$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \xrightarrow{↯} |\psi\rangle = |\alpha|\,|0\rangle + |\beta|\, e^{i\varphi}\,|1\rangle$$

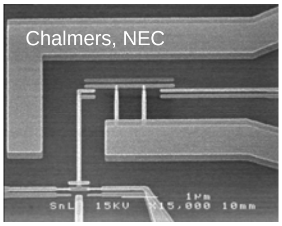
DiVincenzo Criteria for Implementations of a Quantum Computer:

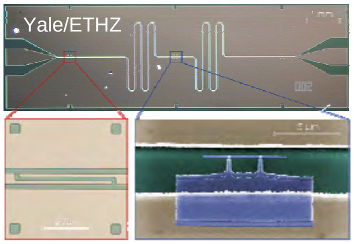
#1. A scalable physical system with well-characterized qubits.
#2. The ability to initialize the state of the qubits to a simple fiducial state.
#3. Long (relative) decoherence times, much longer than the gate-operation time.
#4. A universal set of quantum gates.
#5. A qubit-specific measurement capability.

#6. The ability to interconvert stationary and flying qubits.
#7. The ability to faithfully transmit flying qubits between specified locations.

Quantum Information Processing with Superconducting Circuits







with material from
NIST, UCSB, Berkeley, NEC, NTT, CEA Saclay and Yale

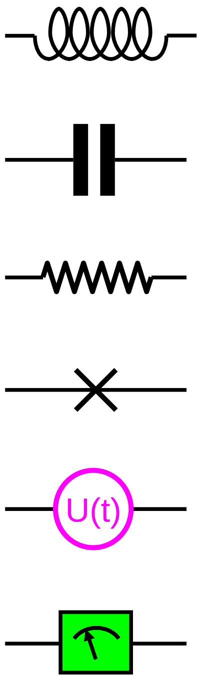
Outline

- construction of quantum mechanical electronic circuits
- the current biased phase qubit
- qubit control
- qubit read-out
- decoherence
- qubit interactions and realizations of gates
- photon/circuit interactions: quantum optics and quantum interfaces
- what's next?

Some Basics ...

… how to construct qubits.

Building Quantum Electrical Circuits



inductor

capacitor

resistor

nonlinear element

voltage source

voltmeters

<u>requirements for quantum circuits:</u>

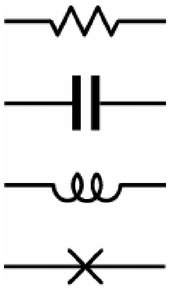
- low dissipation
- non-linear (non-dissipative elements)
- low (thermal) noise

<u>a solution:</u>

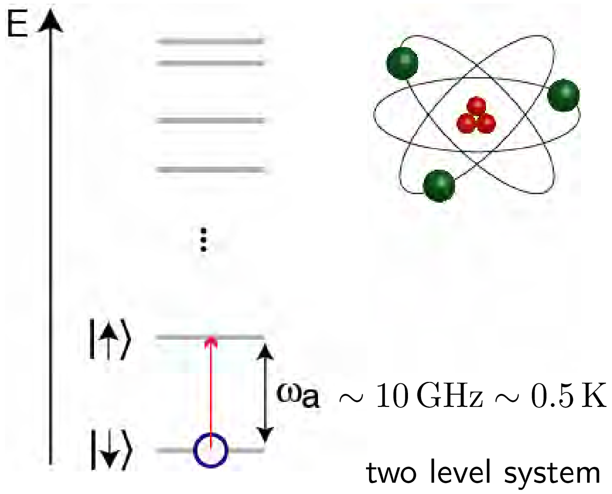
- use superconductors
- use Josephson tunnel junctions
- operate at low temperatures

Building Quantum Electronic Circuits

circuit elements:



macroscopic artificial atom:



$\sim 10\,\mathrm{GHz} \sim 0.5\,\mathrm{K}$

two level system
$\equiv$ qubit

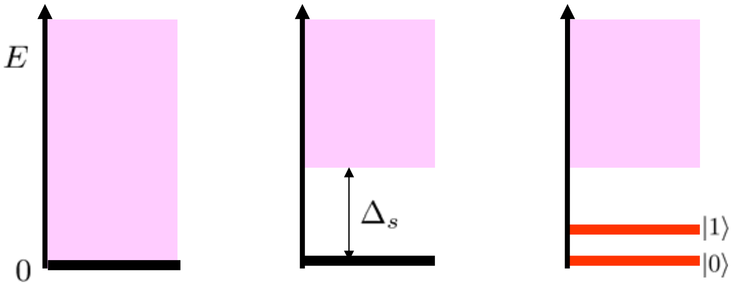
ingredients:

- nonlinearity
- low temperatures
- small dissipation
- isolation from environment

use as basic element for solid state quantum information processor

review article: M. H. Devoret et al., *condmat/0411172* (2004)

Why Superconductors?



normal metal

superconductor

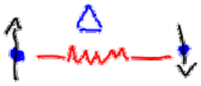
How to make qubit?

- single non-degenerate macroscopic ground state
- elimination of low-energy excitations

Superconducting materials (for electronics):

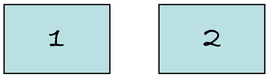
- Niobium (Nb): $2\Delta/h = 725$ GHz, $T_c = 9.2$ K
- Aluminum (Al): $2\Delta/h = 100$ GHz, $T_c = 1.2$ K

Cooper pairs:
bound electron pairs



are Bosons (S=0, L=0)

2 chunks of superconductors

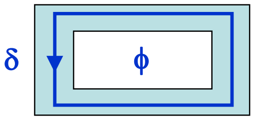


macroscopic wave function

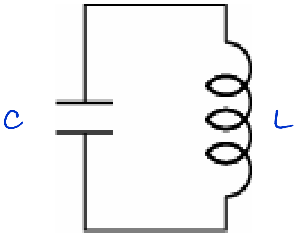
$$\Psi_i = \sqrt{n_i^{CP}}\, e^{i\delta_i}$$

Cooper pair density n_i
and global phase δ_i

phase quantization: $\delta = n\,2\pi$
flux quantization: $\phi = n\,\phi_0$



Superconducting Harmonic Oscillator

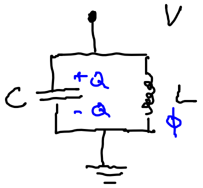


- typical inductor: $L = 1$ nH
- a wire in vacuum has inductance ~ 1 nH/mm
- typical capacitor: $C = 1$ pF
- a capacitor with plate size 10 µm x 10 µm and dielectric AlOx ($\varepsilon = 10$) of thickness 10 nm has a capacitance $C \sim 1$ pF
- resonance frequency

$$\frac{1}{2\pi\sqrt{LC}} \sim 5\,\text{GHz}$$

Quantization of the electrical LC harmonic oscillator:

parallel LC oscillator circuit:



voltage across the oscillator:

$$V = \frac{Q}{C} = -L\frac{\partial I}{\partial t}$$

total energy (Hamiltonian):

$$H = \frac{1}{2}CV^2 + \frac{1}{2}LI^2 = \frac{1}{2}\frac{Q^2}{C} + \frac{1}{2}\frac{\phi^2}{L}$$

with the charge Q stored on the capacitor $Q = VC$

a flux ϕ stored in the inductor $\phi = LI$

properties of Hamiltonian written in variables Q and ϕ:

$$\frac{\partial H}{\partial Q} = \frac{Q}{C} = -L\frac{\partial I}{\partial t} = -\dot{\phi}$$

$$\frac{\partial H}{\partial \phi} = \frac{\phi}{L} = I = \dot{Q}$$

Q and ϕ are canonical variables

see e.g.: Goldstein, Classical Mechanics, Chapter 8, Hamilton Equations of Motion

Quantum version of Hamiltonian

$$\hat{H} = \frac{\hat{Q}^2}{2C} + \frac{\hat{\phi}^2}{2L}$$

with commutation relation

$$[\hat{\phi}, \hat{Q}] = i\hbar$$

compare with particle in a harmonic potential:

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m \omega^2 \hat{x}^2$$

analogy with electrical oscillator:
- charge Q corresponds to momentum p
- flux ϕ corresponds to position x

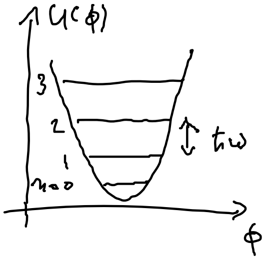
$$[\hat{x}, \hat{p}] = [\hat{x}, -i\hbar \frac{\partial}{\partial x}] = i\hbar$$

$$[\hat{\phi}, \hat{Q}] = [\hat{\phi}, -i\hbar \frac{\partial}{\partial \phi}] = i\hbar$$

Hamiltonian in terms of raising and lowering operators:

$$\hat{H} = \hbar \omega (a^+ a + \frac{1}{2})$$

with oscillator resonance frequency: $\omega = \frac{1}{\sqrt{LC}}$



Raising and lowering operators:

$$a^+ |n\rangle = \sqrt{n+1} \, |n+1\rangle \; ; \; \hat{a} |n\rangle = \sqrt{n} \, |n-1\rangle$$

$$a^+ a |n\rangle = n |n\rangle$$ number operator

in terms of Q and ϕ:

$$\hat{a} = \frac{1}{\sqrt{2\hbar Z_c}} (Z_c \hat{Q} + i \hat{\phi})$$

with Z_c being the characteristic impedance of the oscillator $Z_c = \sqrt{\frac{L}{C}}$

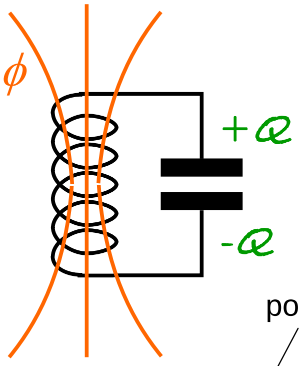
charge Q and flux ϕ operators can be expressed in terms of raising and lowering operators:

$$\hat{Q} = \sqrt{\frac{\hbar}{2 Z_c}} (a + a^+)$$

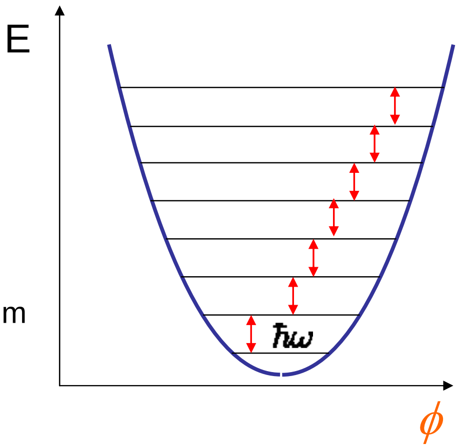
$$\hat{\phi} = \sqrt{\frac{2 Z_c}{\hbar}} (a - a^+)$$

Exercise: Making use of the commutation relations for the charge and flux operators, show that the harmonic oscillator Hamiltonian in terms of the raising and lowering operators is identical to the one in terms of charge and flux operators.

LC Oscillator as a Quantum Circuit



$[\phi, q] = i\hbar$



position momentum

Hamiltonian $H = \frac{\phi^2}{2L} + \frac{q^2}{2C}$

$$\omega = 1/\sqrt{LC}$$

$$H = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)$$

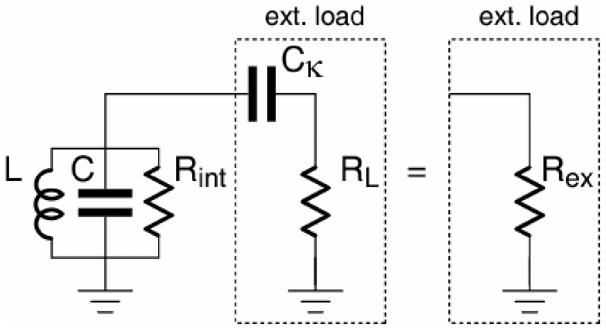
low temperature required:

$$\hbar\omega \gg k_B T$$

1 GHz ~ 50 mK

problem 1: equally spaced energy levels (linearity)

Example: Dissipation in an LC Oscillator



internal losses: R_{int}
conductor, dielectric

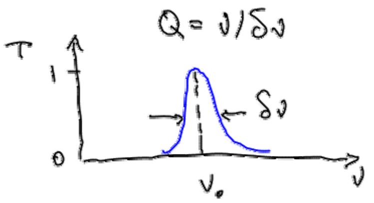
external losses: R_{ext}
radiation, coupling

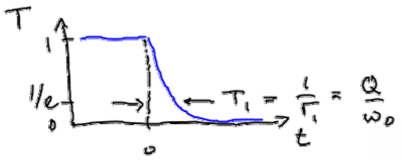
total losses $\frac{1}{R} = \frac{1}{R_{\text{int}}} + \frac{1}{R_{\text{ext}}}$

impedance $Z = \sqrt{\frac{L}{C}}$

quality factor $Q = \frac{R}{Z} = \omega_0 RC$

excited state decay rate $\Gamma_1 = \frac{\omega_0}{Q} = \frac{1}{RC}$



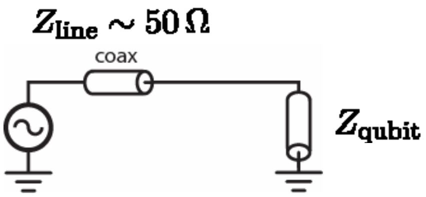


problem 2: avoid internal and external dissipation

Coupling to the Electromagnetic Environment

strong coupling to environment (bias wires):

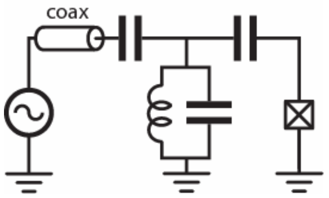
decoherence
from energy relaxation



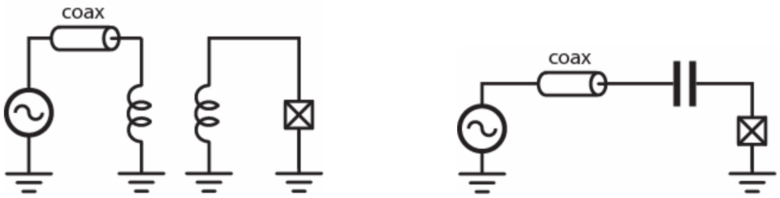
decoupling using impedance transformers:

control decoherence
by circuit design

solution to problem 2



using a resonant circuit



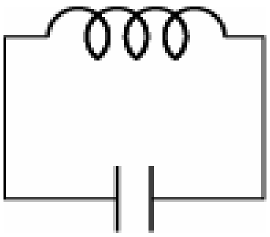
using non-resonant impedance transformers

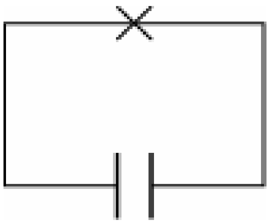
Superconducting Qubits

LC resonator

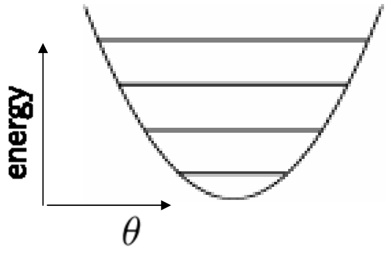
Josephson junction resonator

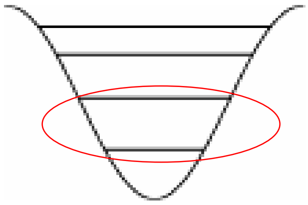
Josephson junction = nonlinear inductor





anharmonicity → effective two-level system

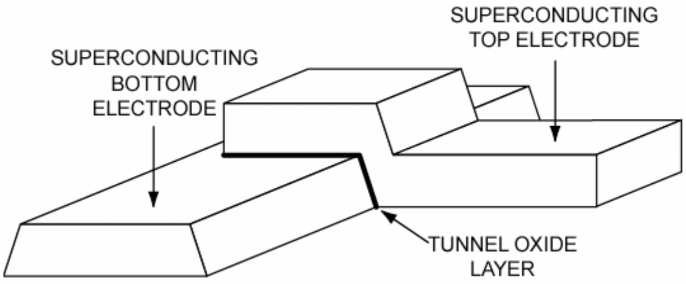




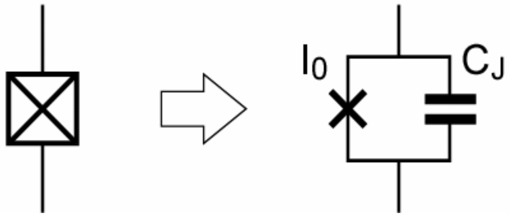
solution to problem 1

A Low-Loss Nonlinear Element

a (superconducting) Josephson junction



- superconductors: Nb, Al
- tunnel barrier: AlO_x

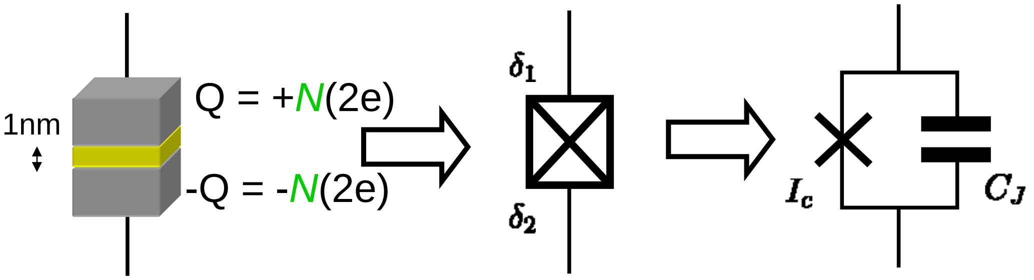


- critical current I_c
- junction capacitance C_J
- high internal resistance R_J

M. Tinkham, *Introduction to Superconductivity* (Krieger, Malabar, 1985).

Josephson Tunnel Junction

the only non-linear LC resonator with no dissipation (BCS, $k_B T \ll \Delta$)



tunnel junction parameters:

- critical current I_c
- junction capacitance C_J
- high internal resistance R_J

Josephson relations:

$$I_0 = I_c \sin\delta$$

$$V = \phi_0 \frac{\partial\delta}{\partial t}$$

flux quantum: $\phi_0 = \frac{h}{2e}$

phase difference: $\delta = \delta_2 - \delta_1$

derivation of Josephson effect, see e.g.: chap. 21 in R. A. Feynman: Quantum mechanics, The Feynman Lectures on Physics. Vol. 3 (Addison-Wesley, 1965)

The Josephson junction as a non-linear inductor

induction law: $V = -L \frac{\partial I}{\partial t}$

Josephson effect: dc-Josephson equation $I = I_c \sin\delta$

$$\frac{\partial I}{\partial t} = I_c \cos\delta \frac{\partial \delta}{\partial t}$$

ac-Josephson equation $V = \frac{\phi_0}{2\pi}\frac{\partial \delta}{\partial t} = \underbrace{\frac{\phi_0}{2\pi I_c}\,\frac{1}{\cos\delta}}_{L_J}\frac{\partial I}{\partial t}$

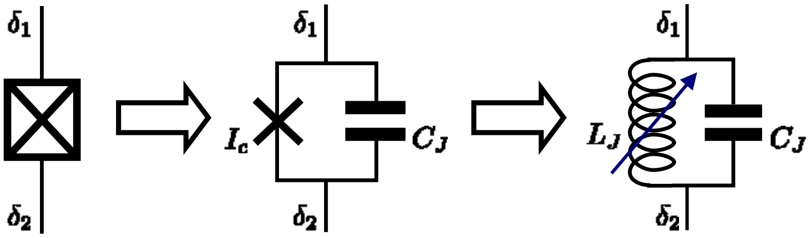
Josephson inductance $L_J = \underbrace{\frac{\phi_0}{2\pi I_c}}_{\text{specific Josephson inductance}}\;\underset{\substack{\uparrow \\ \text{nonlinearity}}}{\frac{1}{\cos\delta}}$

A typical characteristic Josephson inductance for a tunnel junction with $I_c = 100$ nA is $L_{J0} \sim 3$ nH.

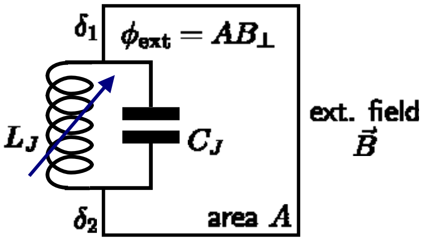
review: M. H. Devoret *et al.*,
Quantum tunneling in condensed media, *North-Holland,* (1992)

The Josephson Junction as a Non-Linear Inductor

a non-linear tunable inductor without dissipation



a non-linear tunable inductor without dissipation

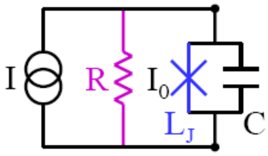


quantization condition for superconducting phase/flux:

$$\delta + \frac{2\pi\phi_{\text{ext}}}{\phi_0} = n2\pi$$

$$\phi_0\frac{\delta}{2\pi} + \phi_{\text{ext}} = n\phi_0$$

Current Biased Phase Qubit



The bias current I distributes into a Josephson current through an ideal Josephson junction with critical current I_c, through a resistor R and into a displacement current over the capacitor C.

Kirchhoff's law:

$$I_b = I_s + I_R + I_C$$
$$= I_c \sin\delta + \frac{V}{R} + C\dot{V}$$

$$I_C = \dot{Q}_C = C\dot{V}$$
$$I_R = V/R$$
$$I_s = I_c \sin\delta$$

use Josephson equations:

$$I_b = I_c \sin\delta + \frac{\phi_0}{2\pi R}\dot{\delta} + \frac{\phi_0 C}{2\pi}\ddot{\delta}$$

W.C. Stewart, Appl. Phys. Lett. **2**, 277, (1968)
D.E. McCumber, J. Appl. Phys. **39**, 3 113 (1968)

looks like equation of motion for a particle with mass m and coordinate δ in an external potential U:

$$m\ddot{\delta} + m\frac{1}{RC}\dot{\delta} + \frac{\partial U(\delta)}{\partial\delta} = 0$$

particle mass: $m = C\,(\phi_0/2\pi)^2$

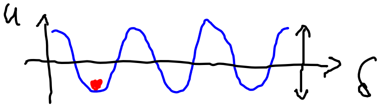
external potential:

$$U(\delta) = \frac{I_c\phi_0}{2\pi}\left(-\frac{I_b}{I_c} - \cos\delta\right)$$

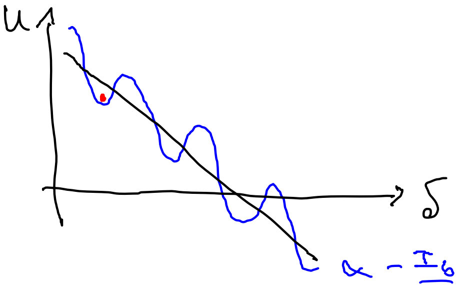
Phase particle in a potential well

$$U(\delta) = \frac{I_c\phi_0}{2\pi}\left(-\frac{I_b}{I_c}\delta - \cos\delta\right) \qquad E_J = \frac{I_c\phi_0}{2\pi}$$

cosine potential for $I_b = 0$:



'tilted washboard' potential for $I_b \neq 0$:



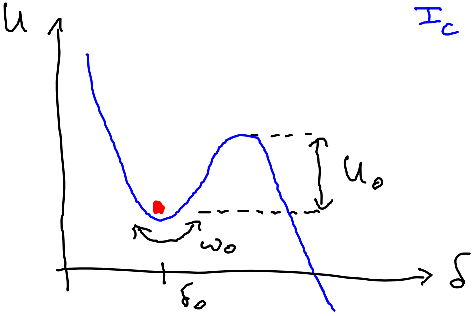
potential barrier:

$$U_0 = 2E_J\left[\sqrt{1-\gamma^2} - \gamma \arccos\gamma\right]$$

oscillation frequency:

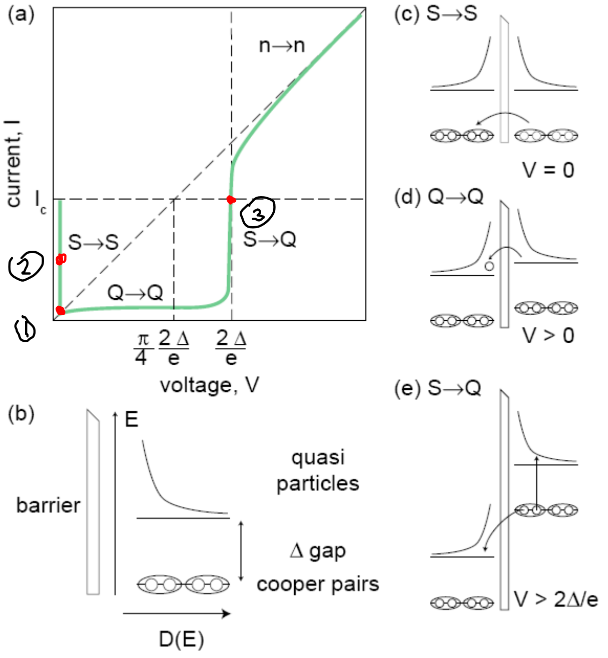
$$\omega_0 = \omega_p(1-\gamma^2)^{1/4} = \sqrt{\frac{U''(\delta_0)}{m}}$$

with: $\gamma = I_b/I_c$; $\omega_p = \sqrt{\frac{2\pi I_c}{\phi_0 C}}$

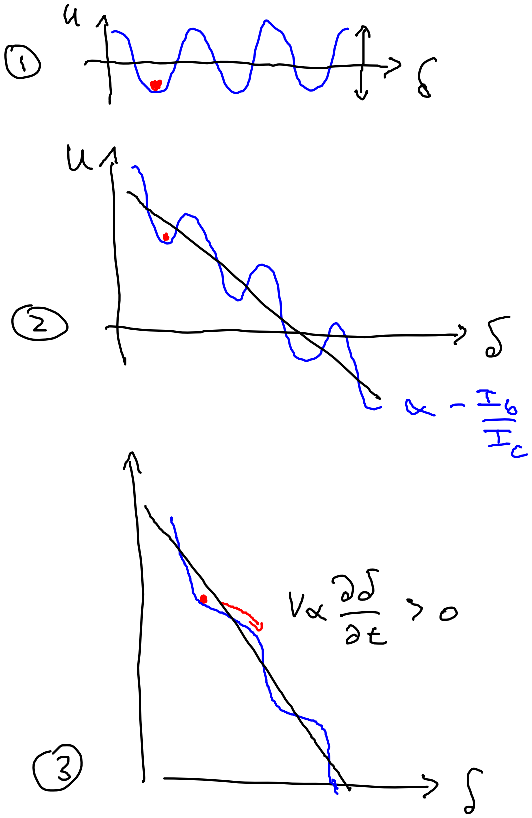


Current-voltage characteristics

typical I-V curve of underdamped Josephson junctions:



band diagram



Thermal Activation and Quantum Tunneling:

bias current dependence

$\omega_0(\gamma)$; $U_0(\gamma)$

thermal activation rate:

$$\Gamma_{th} \approx a_t \frac{\omega_0}{2\pi} \exp\left(-\frac{U_0}{k_B T}\right)$$

damping dependent prefactor

quantum tunneling rate:

$$\Gamma_{qu} = a_q \frac{\omega_0}{2\pi} \exp\left(-\frac{36}{5}\frac{U_0}{\hbar\omega_0}\right)$$

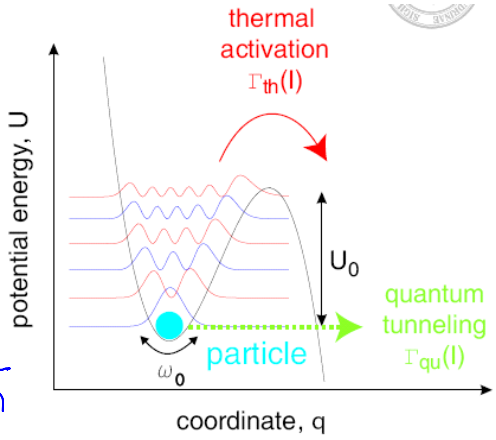
calculated using WKB method (exercise)

$$\Gamma_q \approx a_q \, \omega_0 \exp - \int_{\delta_1}^{\delta_2} \frac{1}{\hbar} \sqrt{2m(U(\delta) - E_0)}$$

energy level quantization:

$$E_n \approx \hbar\omega_0 \left(n + \frac{1}{2}\right)$$

neglecting non-linearity



Quantum Mechanics of a Macroscopic Variable: The Phase Difference of a Josephson Junction
JOHN CLARKE, ANDREW N. CLELAND, MICHEL H. DEVORET, DANIEL ESTEVE, and JOHN M. MARTINIS
Science 26 February 1988 239: 992-997 [DOI: 10.1126/science.239.4843.992] (in Articles) Abstract » References » PDF »

Macroscopic quantum effects in the current-biased Josephson junction
M. H. Devoret, D. Esteve, C. Urbina, J. Martinis, A. Cleland, J. Clarke
in Quantum tunneling in condensed media, North-Holland (1992)

Early Results (1980's)

search for macroscopic quantum effects in superconducting circuits

theoretical predictions:

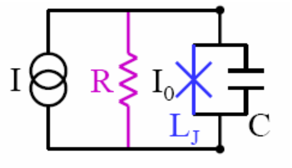
- tunneling ✓
- energy level quantization ✓
- coherence ✗

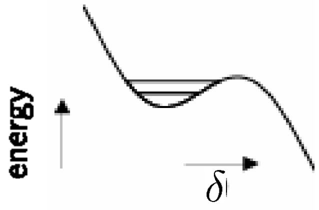
A.J. Leggett *et al.*,
Prog. Theor. Phys. Suppl. **69**, 80 (1980),
Phys. Scr. **T102**, 69 (2002).

short coherence times due to
strong coupling to em environment

experimental verification:

current biased JJ = phase qubit

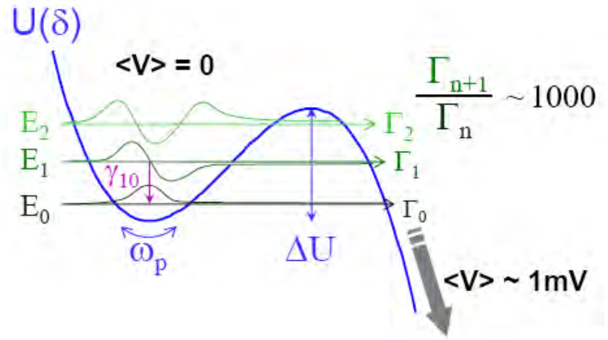


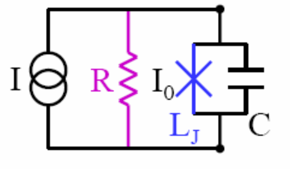


J. Clarke, J. Martinis, M. Devoret *et al.*, *Science* **239**, 992 (1988).

The Current Biased Phase Qubit

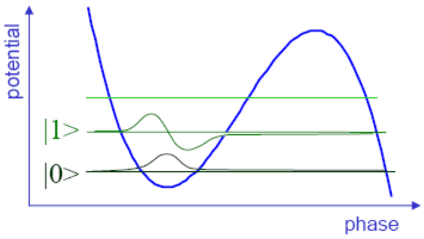
operating a current biased Josephson junction as a superconducting qubit:





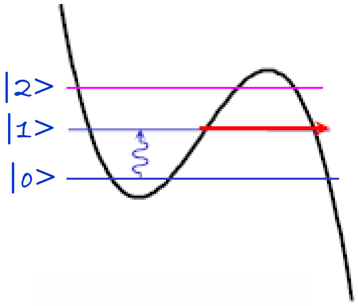
initialization:

wait for |1> to decay to |0>, e.g. by spontaneous emission at rate γ_{10}



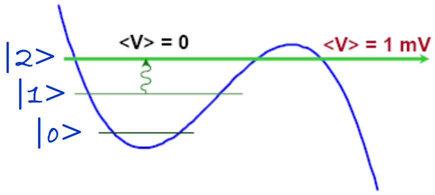
Read-Out

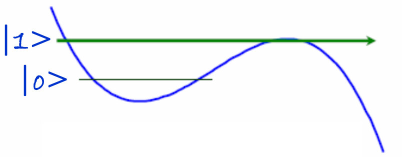
measuring the state of a current biased phase qubit



tunneling:

- prepare state |1> (pump)
- wait ($\Gamma_1 \sim 10^3\,\Gamma_0$)
- detect voltage
- |1> = voltage, |0> = no voltage





pump and probe pulses:

- prepare state |1> (pump)
- drive ω_{21} transition (probe)
- observe tunneling out of |2>

tipping pulse:

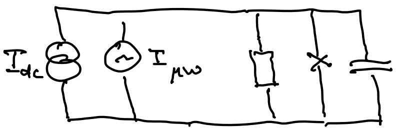
- prepare state |1> (pump)
- drive ω_{21} transition (probe)
- observe tunneling out of |1>

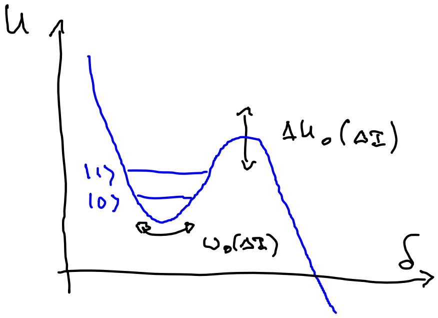
Phase Qubit Control

qubit Hamiltonian:

$$\hat{H} = \frac{\hat{q}^2}{2C} + \frac{\Phi_0}{2\pi}\left(-I_0 \cos\hat{\delta} - (I_{dc} + \Delta I)\,\hat{\delta}\right)$$

$$= \underbrace{\frac{\hat{q}^2}{2C} + \hat{H}_{cu}(I_{dc}, \hat{\delta})}_{} - \frac{\Phi_0}{2\pi}\Delta I\,\hat{\delta}$$





two-level approximation:

$$\hat{H} = \hbar\omega_{01}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \Delta I\,\frac{\Phi_0}{2\pi}\begin{pmatrix} \langle 0|\hat{\delta}|0\rangle & \langle 0|\hat{\delta}|1\rangle \\ \langle 1|\hat{\delta}|0\rangle & \langle 1|\hat{\delta}|1\rangle \end{pmatrix}$$

rotating wave approximation:

$$\tilde{H} = \hat{V}^{+} \hat{H} \hat{V} - i\hbar \hat{V}^{+} \left(\frac{\partial}{\partial t} \hat{V} \right)$$

$$\hat{V} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\omega_{01} t} \end{pmatrix}$$

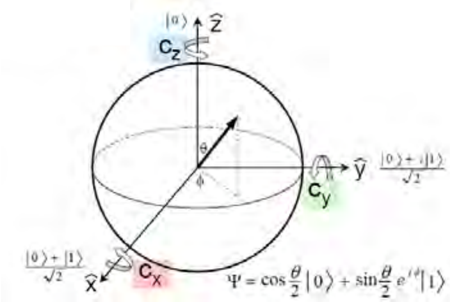
transformed Hamiltonian:

$$\tilde{H} = 0 + \frac{\phi_0}{2\pi} \left(\Delta I_{dc} + I_I \cos \omega_{01} t + I_Q \sin \omega_{01} t \right) \begin{pmatrix} \langle 0|\hat{\delta}|0\rangle & \langle 0|\hat{\delta}|1\rangle e^{-i\omega_{01} t} \\ \langle 1|\hat{\delta}|0\rangle e^{i\omega_{01} t} & \langle 1|\hat{\delta}|1\rangle \end{pmatrix}$$

omit fast oscillating terms:

$$\tilde{H} \approx \frac{\phi_0}{2\pi} \left[\Delta I_{dc} \underbrace{\begin{pmatrix} \langle 0|\hat{\delta}|0\rangle & 0 \\ 0 & \langle 1|\hat{\delta}|1\rangle \end{pmatrix}}_{\hat{\sigma}_z} + \frac{I_I}{2} \underbrace{\begin{pmatrix} 0 & \langle 0|\hat{\delta}|1\rangle \\ \langle 1|\hat{\delta}|0\rangle & 0 \end{pmatrix}}_{\hat{\sigma}_x} + \frac{I_Q}{2} \underbrace{\begin{pmatrix} 0 & i\langle 0|\hat{\delta}|1\rangle \\ i\langle 1|\hat{\delta}|0\rangle & 0 \end{pmatrix}}_{\hat{\sigma}_y} \right]$$

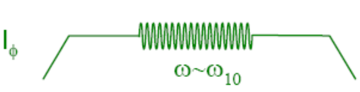
All single bit rotations can be realized using manipulations of the bias current I.

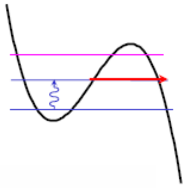


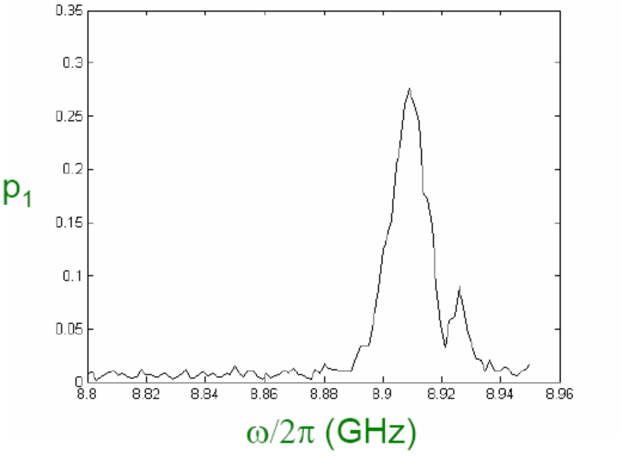
Phase Qubit Control (I): Spectroscopy

spectroscopy

- apply long ($\Delta t > 1/\Gamma_1 = T_1$) resonant ($\omega = \omega_{01}$) microwave pulse to qubit
- qubit will be with equal probability ½ in states |0> and |1>
- measure qubit state and determine excited state probability P_1







courtesy UCSB/NIST

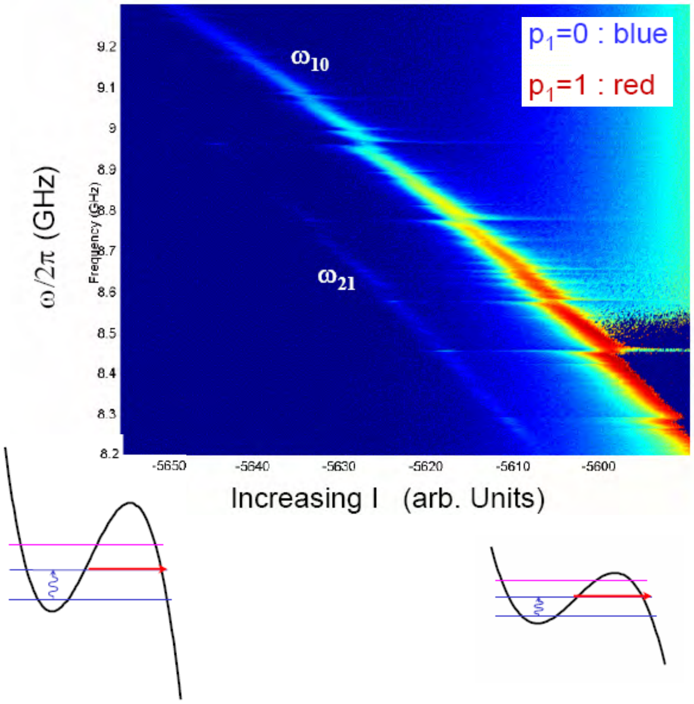
Tuning Energy Levels

phase qubit spectroscopy

- vary level separation using bias current I_{dc}

$$\omega_{01} = \omega_p \left(1 - \left(\frac{I_{dc}}{I_c}\right)^2\right)^{1/4}$$

☐ ω_{21} transition observable (indication for non-zero temperature)

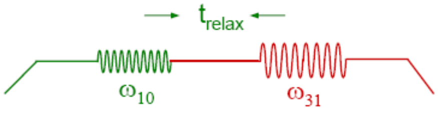


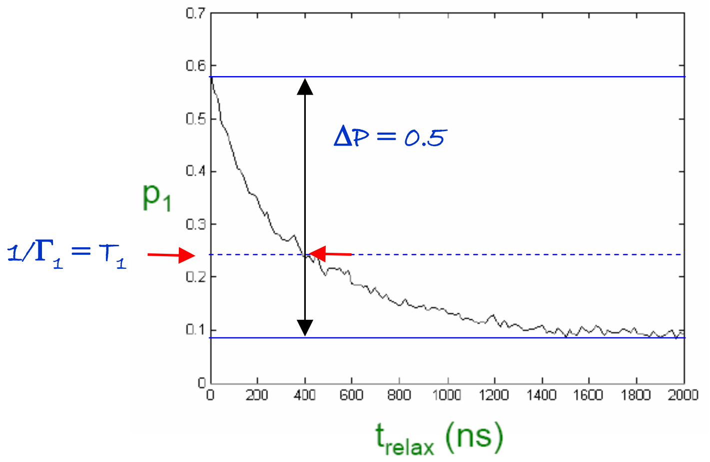
courtesy UCSB/NIST

Energy Relaxation: T$_1$ Measurement

relaxation measurement:

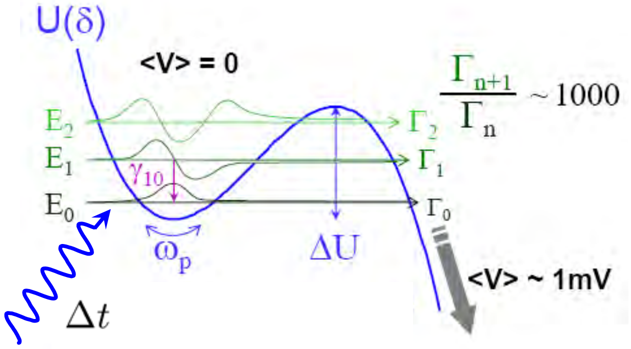
- apply long ($\Delta t > 1/\Gamma_1 = T_1$) resonant ($\omega = \omega_{01}$) microwave pulse to qubit
- qubit will be with equal probability ½ in states |0> and |1>
- vary waiting time t_{relax}
- measure qubit state and determine excited state probability P_1



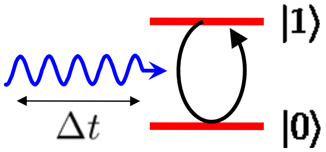


courtesy UCSB/NIST

Coherent Manipulation of a Phase Qubit



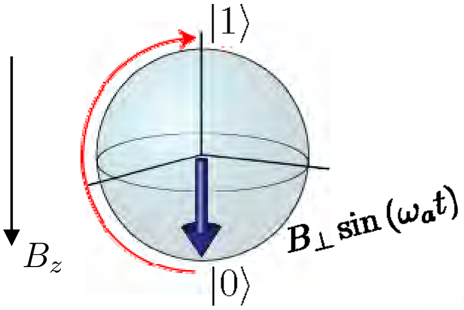
coherent manipulation



single qubit operations:

- apply short ($\Delta t < 1/\Gamma_1 = T_1$) resonant ($\omega = \omega_{01}$) microwave pulse to qubit
- vary pulse length Δt or pulse amplitude (I_q)
- measure qubit state

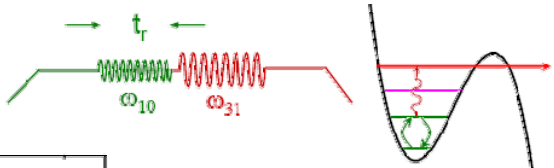
Bloch sphere picture

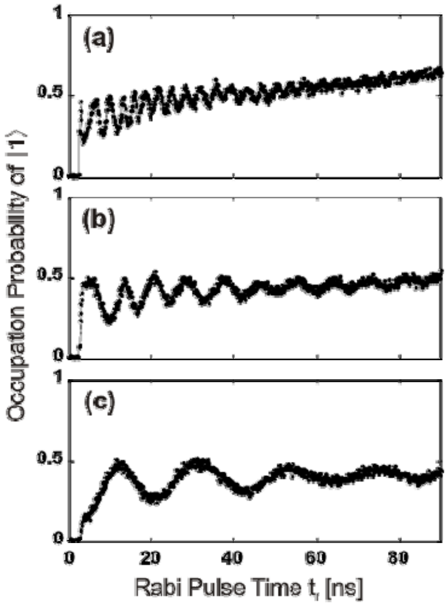


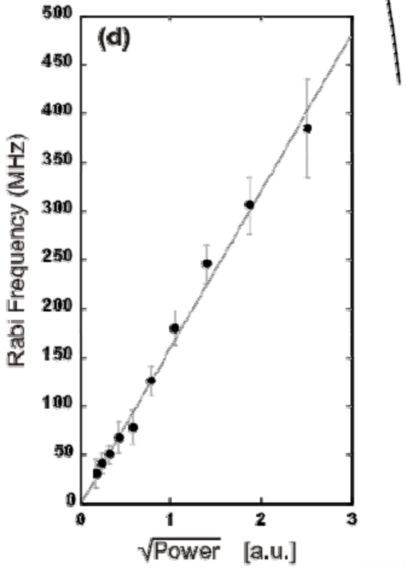
Rabi Oscillations

coherent single qubit manipulation:

- qubit rotation angle proportional to pulse length Δt
- qubit rotation frequency prop. to drive amplitude I_q [equiv. $(\text{power})^{1/2}$]





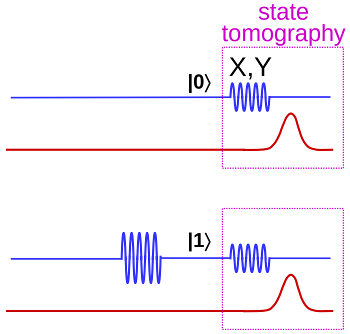


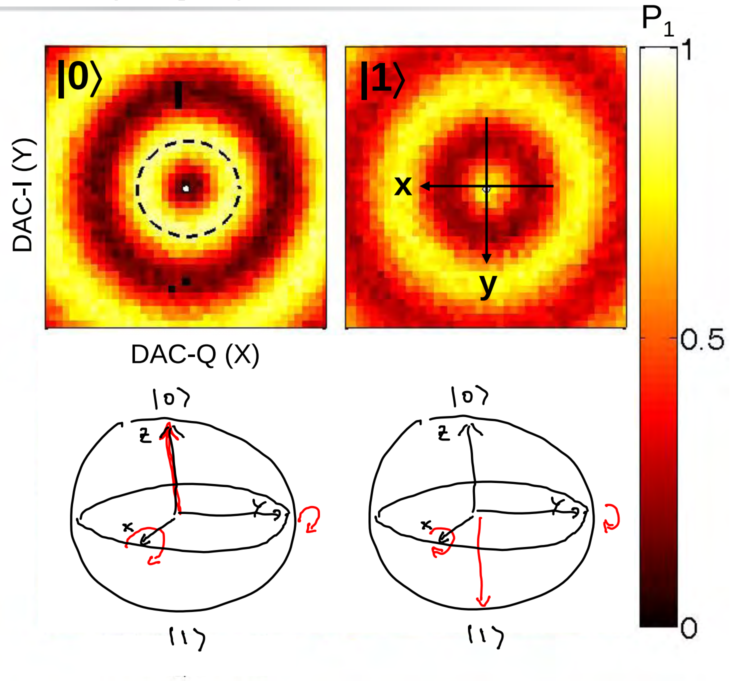
here:

- poor measurement
- or poor state prep.

courtesy UCSB/NIST

Phase Qubit State Tomography



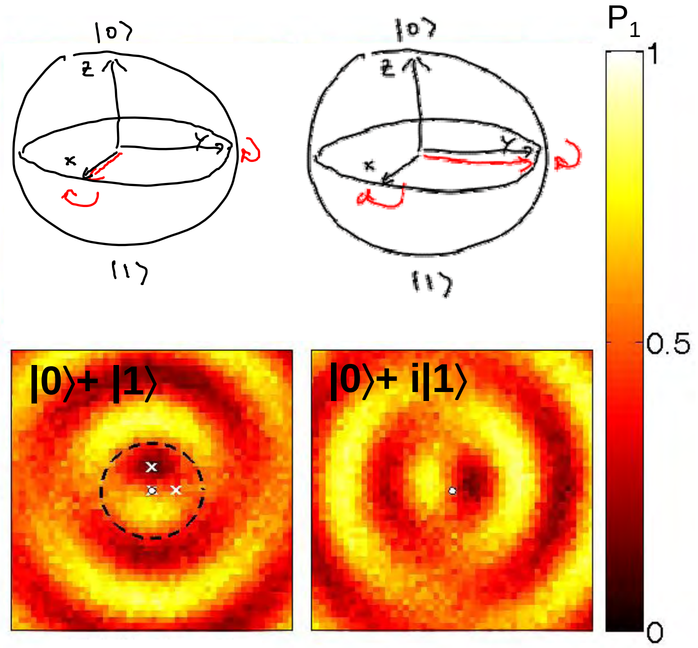


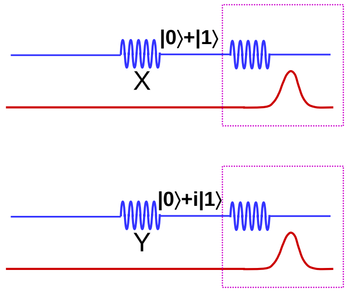
- preparation of initial state (computational basis state)
- X or Y rotation
- measure qubit state along Z

courtesy UCSB/NIST

Phase Qubit State Tomography

- preparation of initial state (here a superposition state)
- X or Y rotation
- measure qubit state along Z

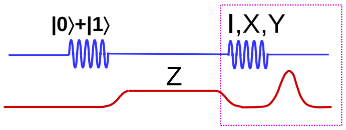




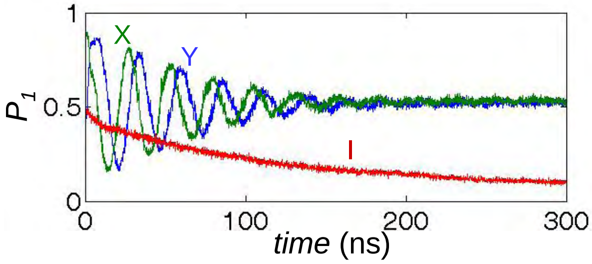
courtesy UCSB/NIST

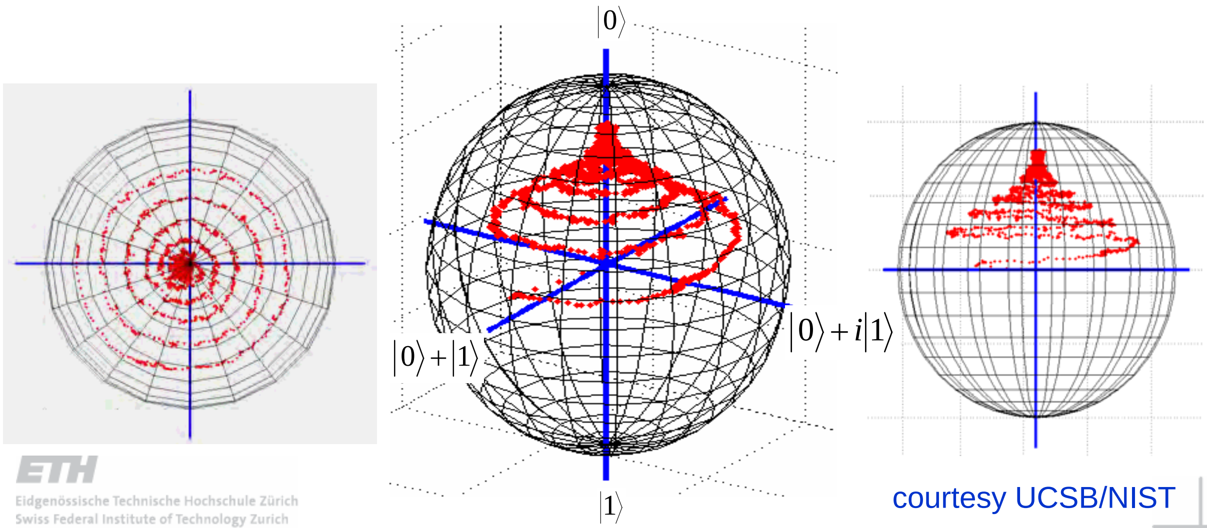
- sufficient information to fully reconstruct qubit state
- i.e. angles θ and ϕ of state vector on Bloch sphere, amplitude is coherence dependent

Tracking the Qubit State Vector



- variable delay before tomography

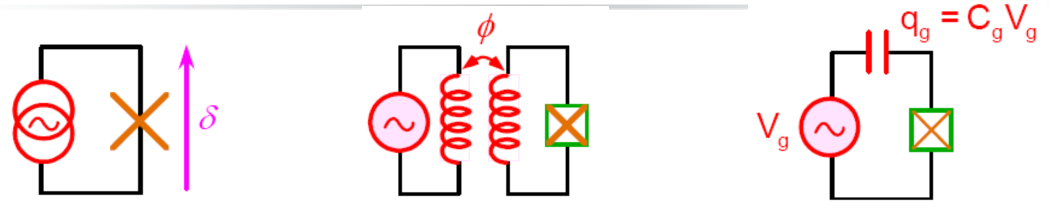




ETH
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

courtesy UCSB/NIST

Josephson Junctions in Different Bias Circuits



current bias	flux bias	charge bias
discussed previously	to be discussed in student presentations	now !

commutation relation: $[\hat{\phi}, \hat{Q}] = i\hbar$

with $\hat{\phi} = \phi_0 \frac{\hat{\delta}}{2\pi}$ flux ϕ in terms of phase difference δ

$\hat{Q} = 2e\hat{N}$ charge Q in terms of number of charges N

phase - number commutation relation: $[\hat{\delta}, \hat{N}] = i$

Cooper pair box Hamiltonian:

Hamiltonian: $$\hat{H} = \underbrace{E_c(\hat{N} - N_g)^2}_{\text{electrostatic}} - \underbrace{E_J \cos\hat{\delta}}_{\text{magnetic energy}} = \frac{E_J}{2}\left(e^{i\hat{\delta}} + e^{-i\hat{\delta}}\right)$$

charging energy Josephson coupling Energy

$$E_c = \frac{(2e)^2}{2\,C_\Sigma} \qquad E_J = \frac{\phi I_c}{2\pi}$$

Hamiltonian in charge representation:

$$\hat{H} = E_c(N-N_g)^2 |N\rangle\langle N| - \frac{E_J}{2}\sum_N \left(|N+1\rangle\langle N| + |N\rangle\langle N+1|\right)$$

easy to diagonalize numerically

$$\hat{H} = \begin{pmatrix} \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & E_c(-1-N_g)^2 & -E_J/2 & 0 & \cdots \\ \cdots & -E_J/2 & E_c(0-N_g)^2 & -E_J/2 & \cdots \\ \cdots & 0 & -E_J/2 & E_c(1-N_g)^2 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

relation between **phase** and **number basis**:

$$|\delta\rangle = \frac{1}{\sqrt{2\pi}}\sum_N e^{i\hat{N}\delta}|N\rangle \qquad \text{with} \quad e^{i\hat{\delta}}|N\rangle = |N+1\rangle$$

Phase representation of Cooper pair box Hamiltonian:

$$\hat{H} = E_c(\hat{N} - N_g)^2 - E_J\cos\hat{\delta} \qquad \text{with} \quad \hat{N} = \frac{\hat{Q}}{2e} = -i\hbar\frac{1}{2e}\frac{\partial}{\partial\phi}$$

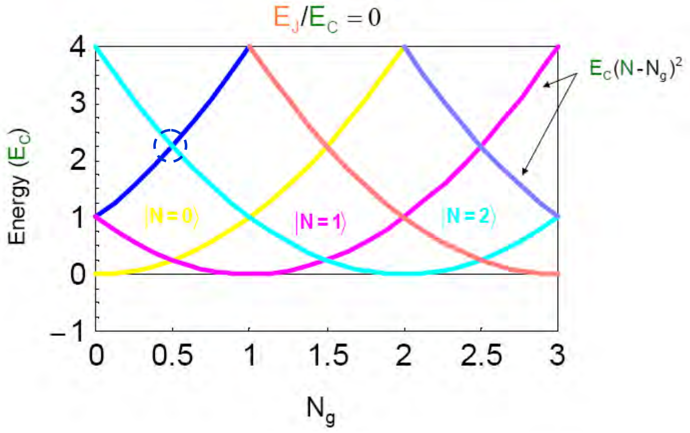
$$= E_c\left(-i\frac{\partial}{\partial\delta} - N_g\right)^2 - E_J\cos\hat{\delta} \qquad = -i\frac{\partial}{\partial\delta}$$

Equivalent solution to the Hamiltonian can be found in both representations, e.g. by numerically solving the Schrödinger equation for the charge (N) representation or analytically solving the Schrödinger equation for the phase (δ) representation.

$$\hat{H}|\psi\rangle = E|\psi\rangle$$

solutions for $E_J = 0$:

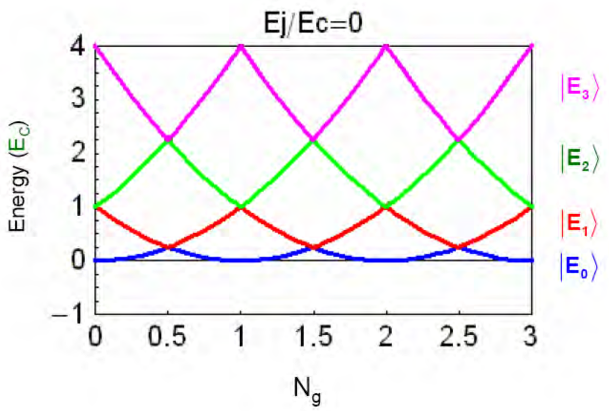
- crossing points are charge degeneracy points



Energy Levels

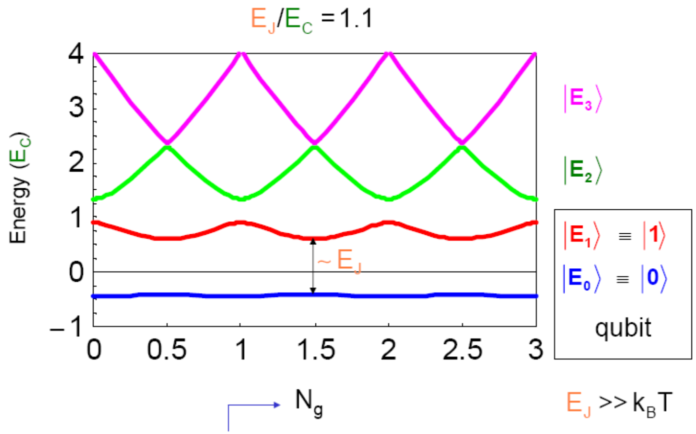
energy levels for finite $E_J=0$:

- energy bands are formed
- bands are periodic in N_g



energy bands for finite E_J

- E_J scales level separation at charge degeneracy

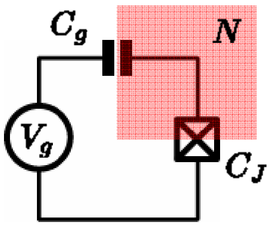


tunable artificial atom

$E_J >> k_BT$

Charge Qubits

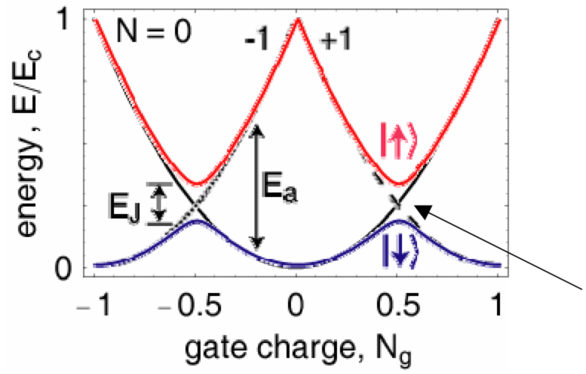
Cooper pair box



charging energy $$E_C = \frac{(2e)^2}{2(C_J+C_g)}$$

gate charge $$N_g = \frac{C_g V_g}{2e}$$

Josepshon energy $$E_J = \frac{I_0 \Phi_0}{2\pi} = \frac{h\Delta}{8e^2 R_J}$$



$$H = E_C (N - N_g)^2 - E_J \cos\theta$$

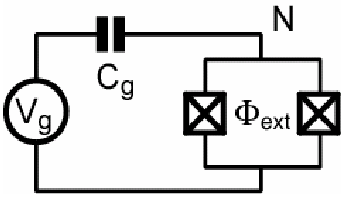
electrostatic energy Josephson energy

charge degeneracy

V. Bouchiat, D. Vion, P. Joyez, D. Esteve, and M. H. Devoret,
Physica Scripta **T76,** 165 (1998).

Tuning the Josephson Energy

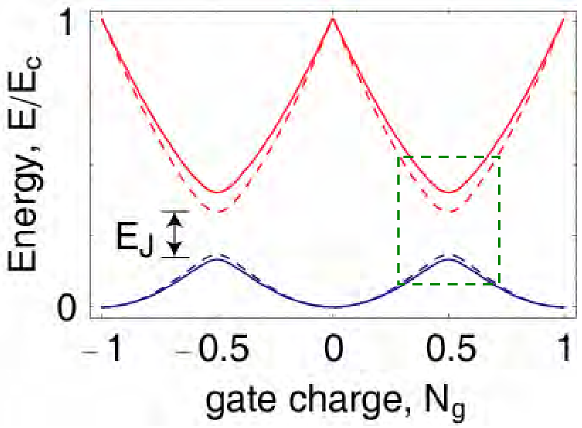
split Cooper pair box in perpendicular field



$$H = E_C\,(N - N_g)^2 - E_{J,\max}\cos\left(\pi\,\frac{\Phi_{\rm ext}}{\Phi_0}\right)$$

SQUID modulation of Josephson energy

$$E_J = E_{J,\max}\cos\left(\pi\,\frac{\Phi_{\rm ext}}{\Phi_0}\right)$$



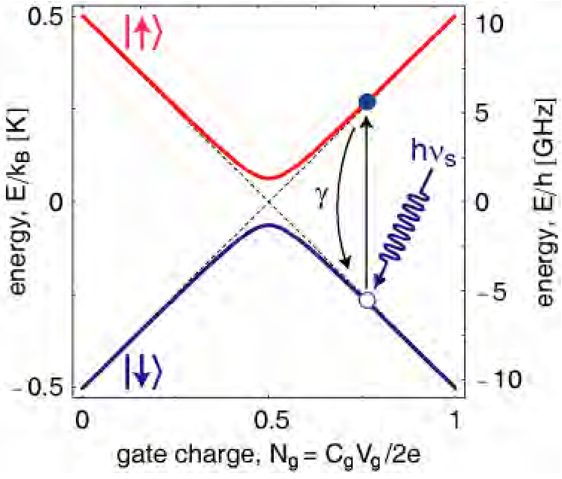
consider two state approximation

J. Clarke, *Proc. IEEE* **77**, 1208 (1989)

Two-State Approximation

2-state Hamiltonian and level separation:

$$H = -1/2\,(E_{\rm el}\,\sigma_x + E_J\,\sigma_z)$$
$$E = \sqrt{E_{\rm el}^2 + E_J^2}$$



in-situ controllable parameters:

$$E_{\rm el} = E_C\,(1/2 - N_g)$$
$$E_J = E_{J,\max}\cos(\pi\Phi_{\rm ext}/\Phi_0)$$

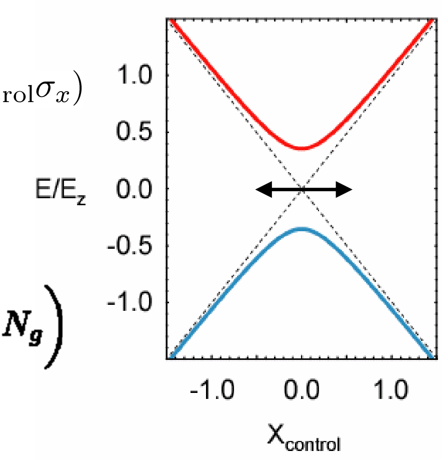
E_C, $E_{J,\max}$ **engineerable in fabrication**

Nakamura, Pashkin, Tsai *et al. Nature* **398**, **421**, **425** (1999,2003, 2003)

Control of Charge Qubit

effective Hamiltonian

$$H_{\text{qubit}} = -E_z\left(\sigma_z + X_{\text{control}}\sigma_x\right)$$

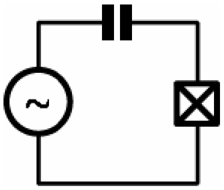


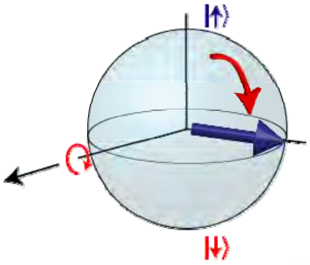
energy splitting $$E_z = \frac{E_J}{2}$$

control parameter $$X_{\text{control}} = 2\frac{E_C}{E_J}\left(\frac{1}{2} - N_g\right)$$

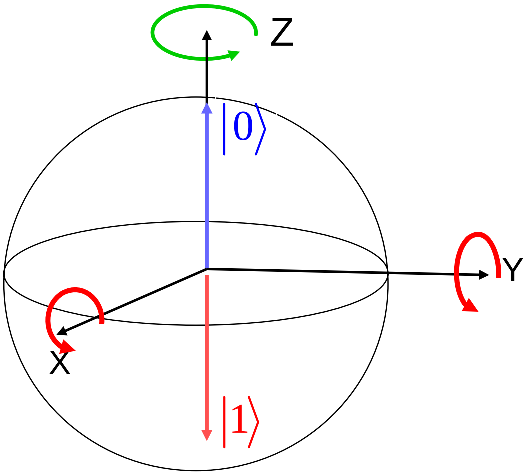
gate charge $$N_g = \frac{C_g V_g}{2e}$$

experimental implementation

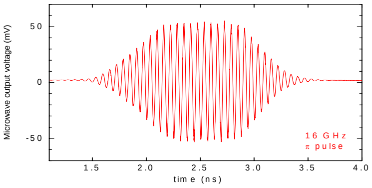




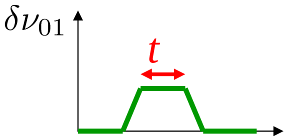
Implementation of Single Qubit Rotations



x,y rotations by microwave pulses



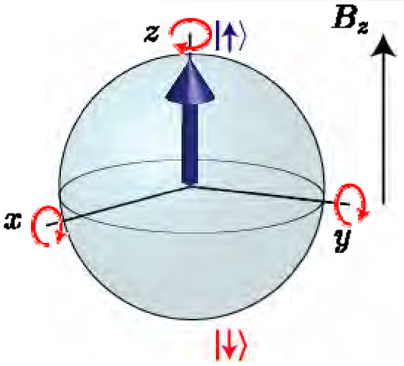
z rotations by adiabatic pulses



full control over quantum state of two level system !

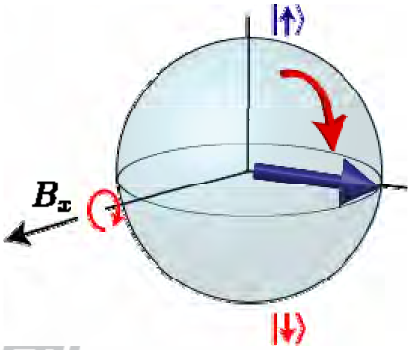
E. Collin *et al.* *Phys. Rev. Lett.* **93**, 157005 (2004)

Single Qubit Control



Bloch sphere representation of single qubit manipulation

rotations about x, y, z axes

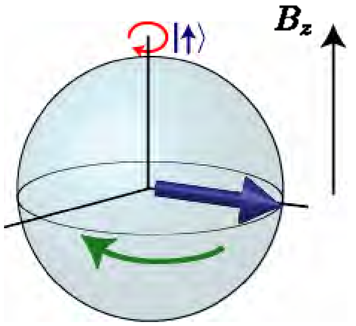


change qubit state with perpendicular field $B \perp B_z$

x-rotation

y-rotation is equivalent

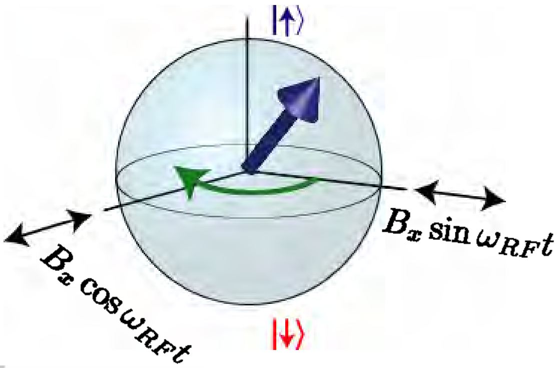
Larmor Precession



spin precession frequency

$$\hbar\omega_{01} = \mu B$$

$\equiv$ qubit transition frequency



rotating frame:

field oscillating at $\omega_{RF} \sim \omega_{01}$ drives transitions

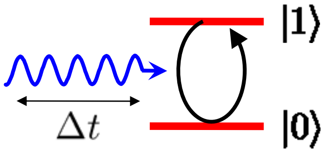
at $\omega_{01} = \omega_{RF}$ driving field looks static in rotating frame

Coherent Control

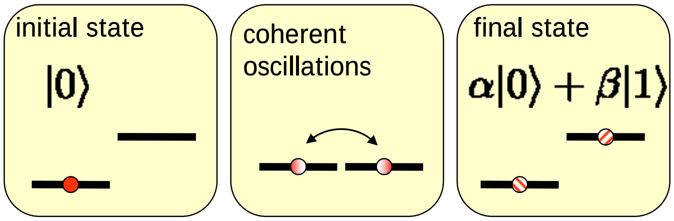
two approaches:

- increase coherence times

resonant microwave pulse



- use ultra-fast manipulation



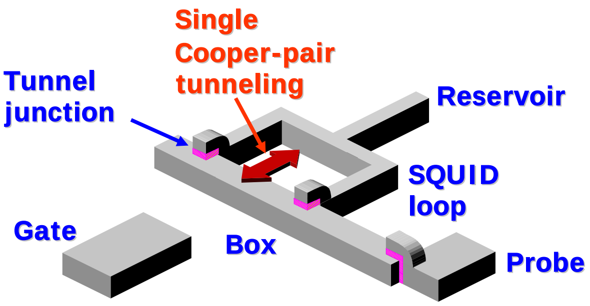
initial state $|0\rangle$

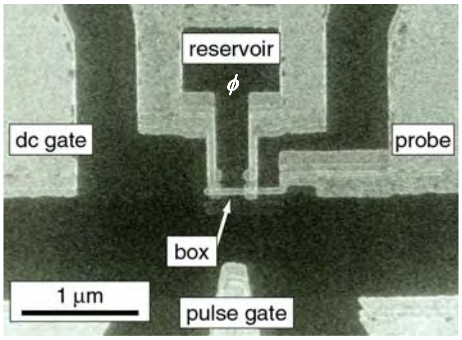
coherent oscillations

final state $\alpha|0\rangle + \beta|1\rangle$

fast dc-pulse

The First Solid State Qubit - Ever

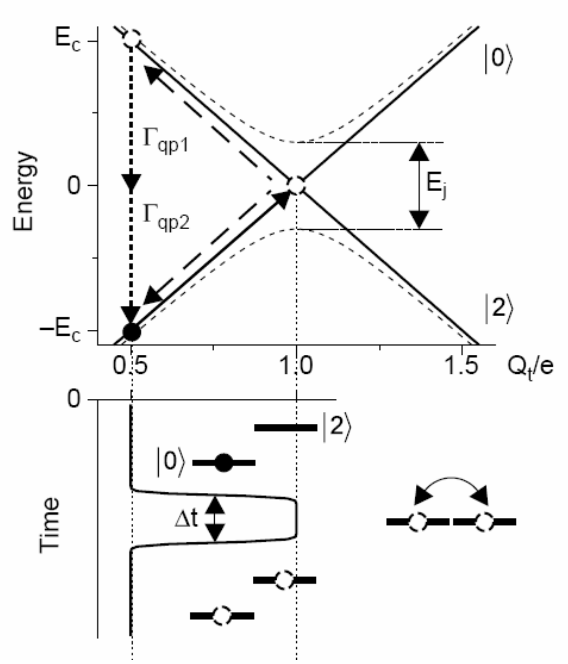


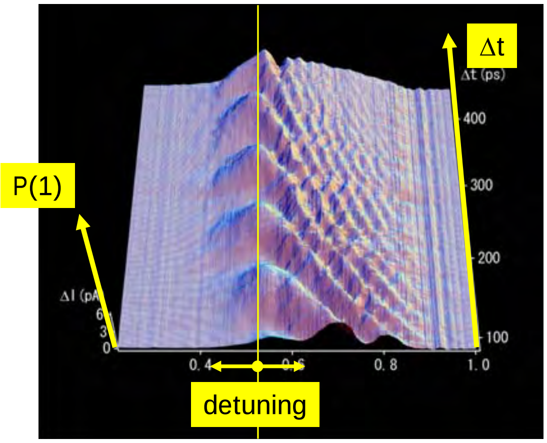


Y. Nakamura *et al.*, *Nature* **398**, 786 (1999)

- Cooper pair box with probe junction for measurement
- read out by measurement of average current
- first successful experiment

Rabi Oscillations





Y. Nakamura *et al.*, *Nature* **398**, 786 (1999)

- picosecond fast pulses enable first time-resolved measurement despite short coherence time
- start of superconducting quantum information processing

Superconducting Qubits in Microwave Resonators

Why to put a qubit into a resonator?

- strong qubit - light interactions
- coherent exchange of single quanta (photons) between a qubit and a cavity

benefits for quantum computation:

- communication between qubits via photons
- non-local qubit/qubit gate interactions
- protection of qubits from spontaneous emission
- quantum non-demolition qubit read-out

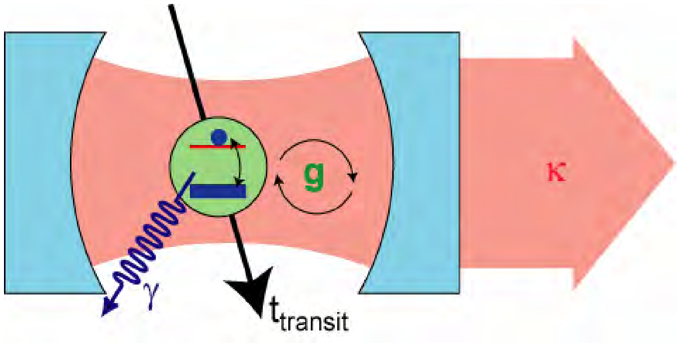
other interesting aspects:

- quantum optics experiments with circuits
- cavity quantum electrodynamics
- single photon generation and detection

Outline

- What is cavity quantum electrodynamics?
- How to realize it with superconducting circuits
- demonstration of the concept (vacuum Rabi oscillations)
- applications for quantum information processing

Cavity Quantum Electrodynamics (CQED)



Jaynes-Cummings Hamiltonian

$$H = \hbar\omega_r \left(a^\dagger a + \frac{1}{2} \right) + \frac{\hbar\omega_a}{2}\sigma^z + \hbar g(a^\dagger\sigma^- + a\sigma^+) + H_\kappa + H_\gamma$$

strong coupling limit $(g = dE_0/\hbar > \gamma,\, \kappa\, , 1/t_{\text{transit}})$

D. Walls, G. Milburn, Quantum Optics (Spinger-Verlag, Berlin, 1994)

Dressed States Energy Level Diagram

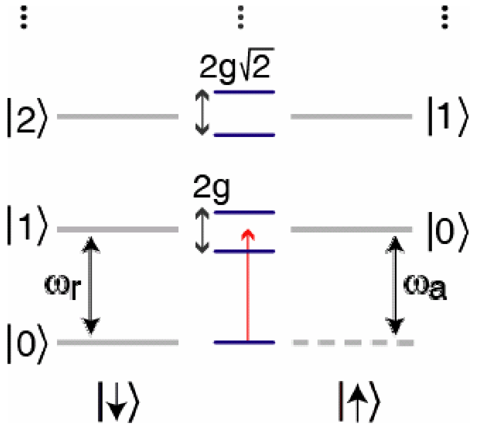
$$H = \hbar\omega_r \left(a^\dagger a + \frac{1}{2} \right) + \frac{\hbar\omega_a}{2}\sigma^z + \hbar g(a^\dagger\sigma^- + a\sigma^+)$$

in resonance:

$$\omega_a - \omega_r = \Delta = 0$$

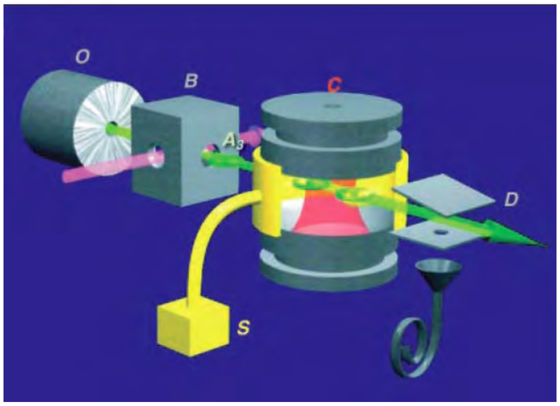
strong coupling limit:

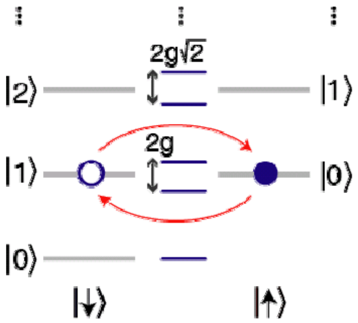
$$g = \frac{dE_0}{\hbar} > \gamma,\, \kappa$$

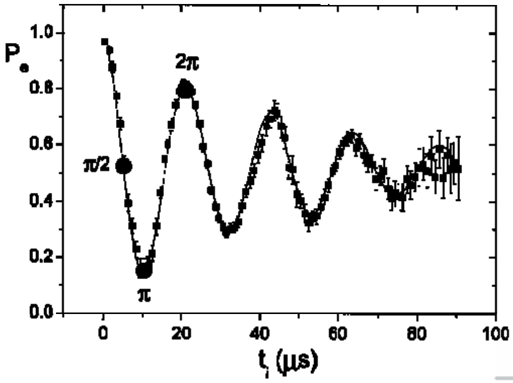


Atomic cavity quantum electrodynamics reviews:
H. Mabuchi, A. C. Doherty *Science* **298**, 1372 (2002)
J. M. Raimond, M. Brune, & S. Haroche *Rev. Mod. Phys.* **73**, 565 (2001)

Vacuum Rabi Oscillations with Rydberg Atoms

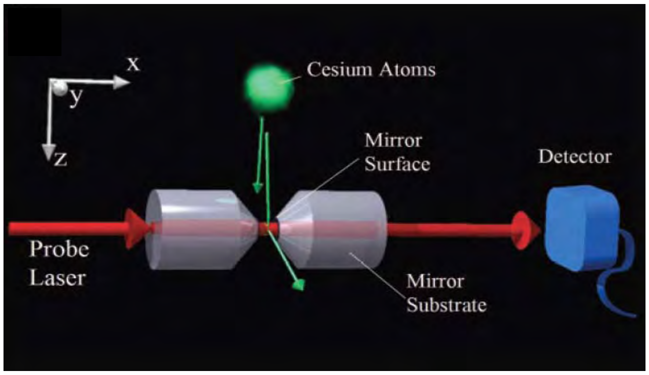


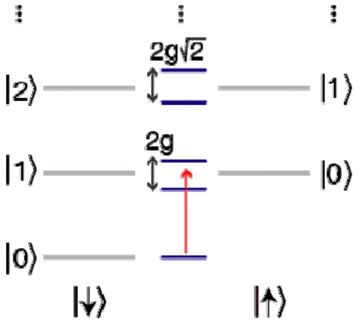


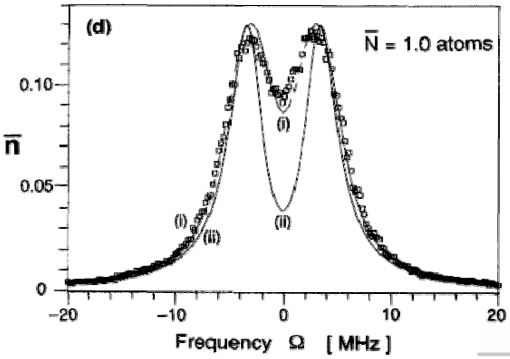


Review: J. M. Raimond, M. Brune, and S. Haroche
Rev. Mod. Phys. **73**, 565 (2001)
P. Hyafil, ..., J. M. Raimond, and S. Haroche,
Phys. Rev. Lett. **93**, 103001 (2004)

Vacuum Rabi Mode Splitting with Alkali Atoms

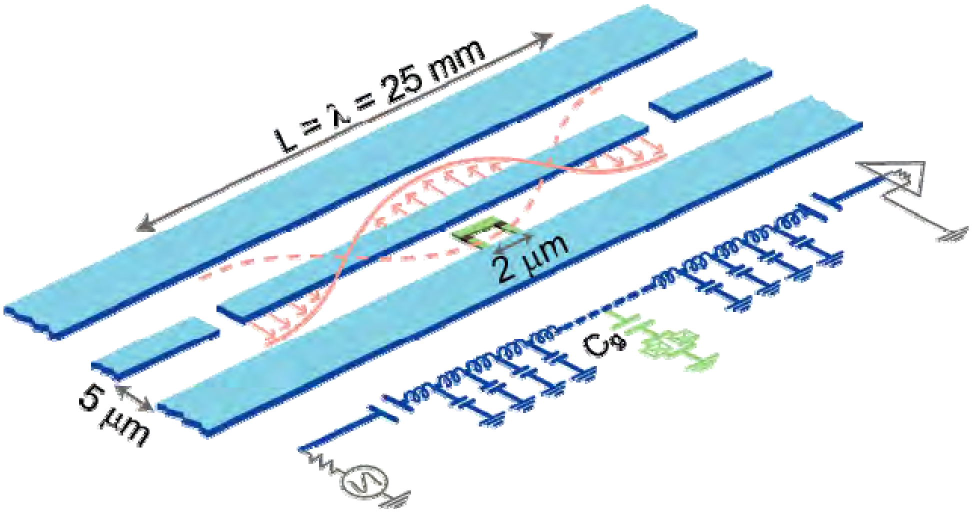






R. J. Thompson, G. Rempe, & H. J. Kimble,
Phys. Rev. Lett. **68** 1132 (1992)
A. Boca, ... , J. McKeever, & H. J. Kimble
Phys. Rev. Lett. **93**, 233603 (2004)

Circuit QED Architecture

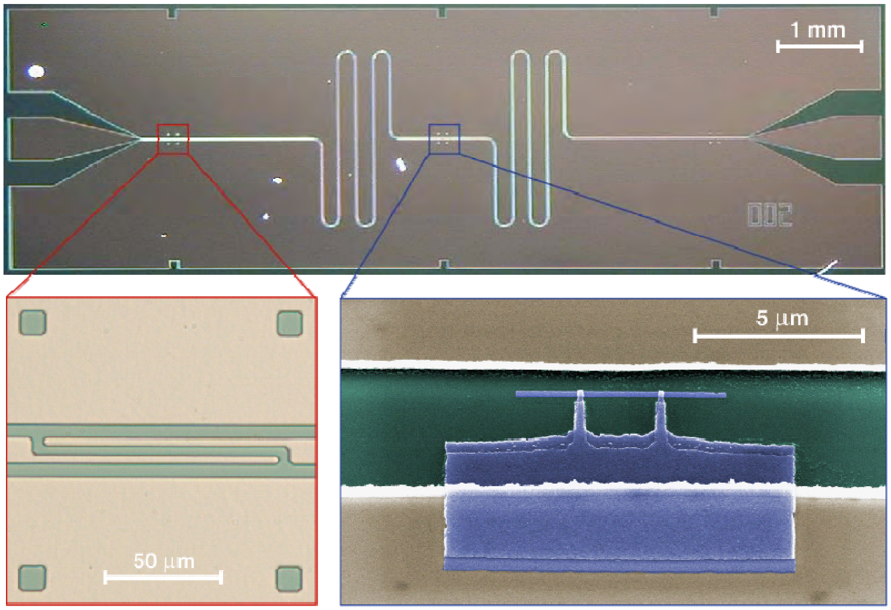


elements

- the cavity: a superconducting 1D transmission line resonator (large E_0)
- the artificial atom: a Cooper pair box (large d)

A. Blais, R.-S. Huang, A. Wallraff, S. M. Girvin, and R. J. Schoelkopf,
PRA **69,** 062320 (2004)

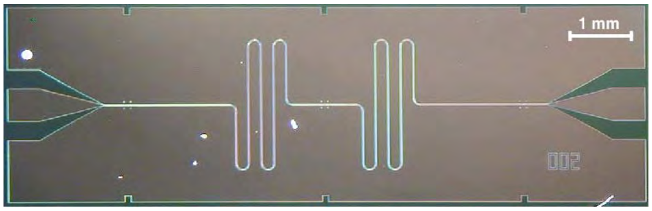
Realization



superconducting cavity QED circuit

A. Wallraff, D. Schuster, ..., S. Girvin, and R. J. Schoelkopf,
Nature (London) **431**, 162 (2004)

Vacuum Field in 1D Cavity



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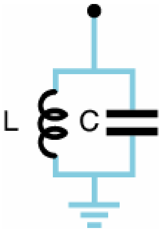


voltage across resonator in vacuum state ($n = 0$)

harmonic oscillator

$$V_{0,\mathrm{rms}} = \sqrt{\frac{\hbar\omega_r}{2C}} \approx 1\,\mu\mathrm{V}$$

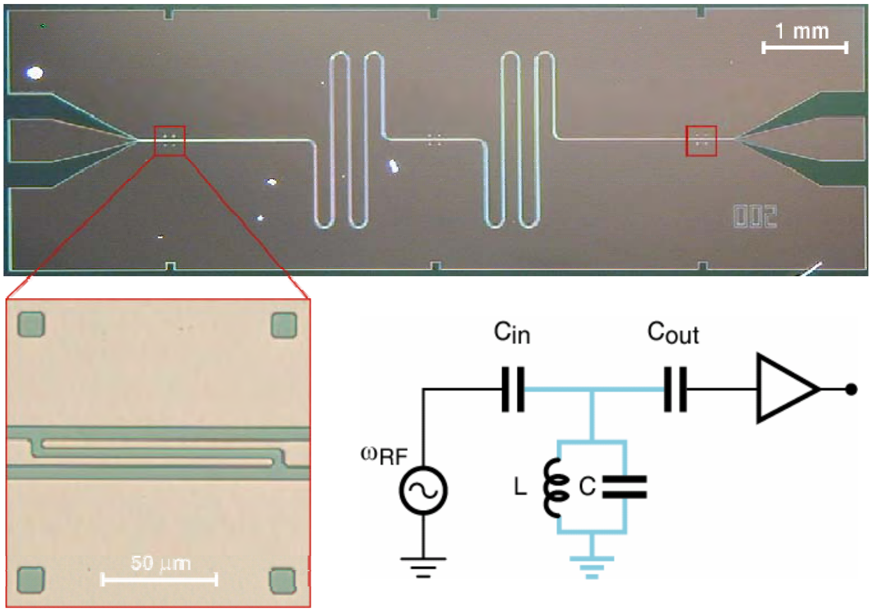
$$H_r = \hbar\omega_r\left(a^\dagger a + \frac{1}{2}\right)$$

$$E_0 = \frac{V_{0,\mathrm{rms}}}{b} \approx 0.2\,\mathrm{V/m}$$

×100 larger than E_0 in 3D microwave cavity

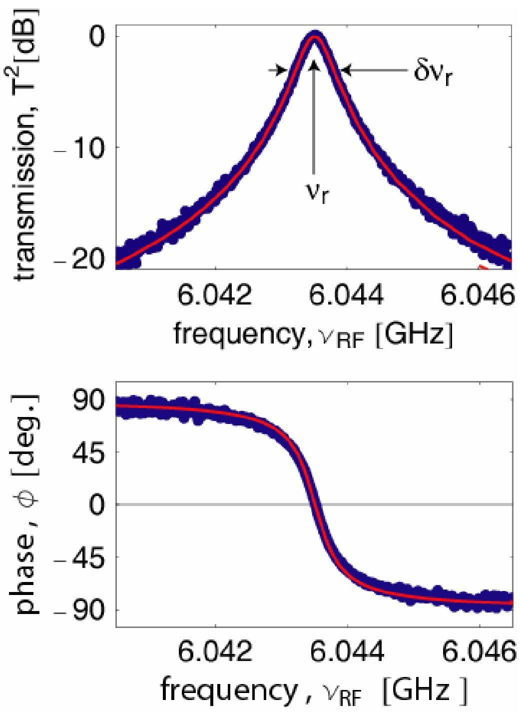
for $\omega_r/2\pi \approx 6\,\mathrm{GHz}$ ($C \sim 1\,\mathrm{pF}$), $b \approx 5\,\mu\mathrm{m}$

Cavity Properties



- photon lifetime (quality factor) controlled by coupling $C_{\mathrm{in/out}}$

Resonator Quality Factor and Photon Lifetime



resonance frequency:

$$\nu_r = 6.04\,\mathrm{GHz}$$

quality factor:

$$Q = \frac{\nu_r}{\delta\nu_r} \approx 10^4$$

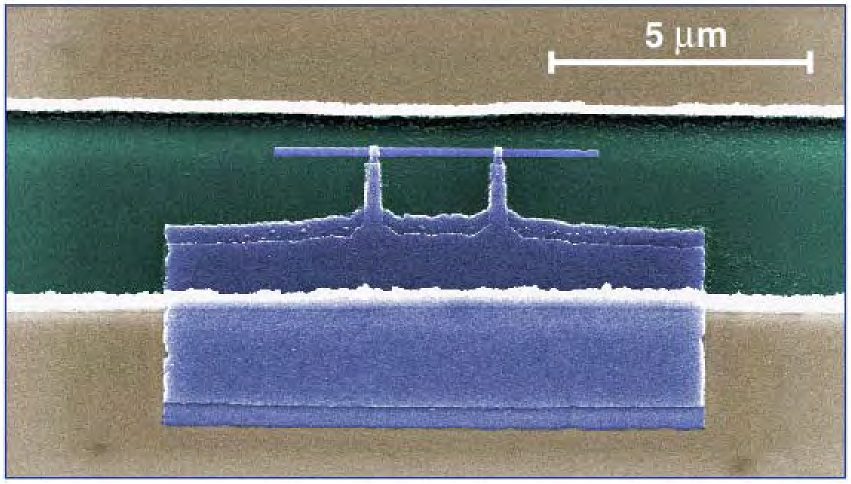
photon decay rate:

$$\frac{\kappa}{2\pi} = \frac{\nu_r}{Q} \approx 0.8\,\mathrm{MHz}$$

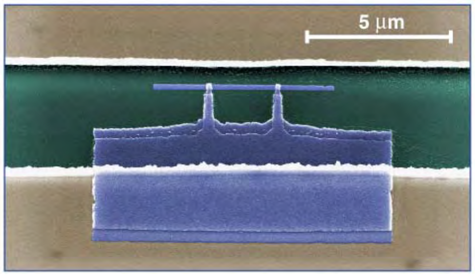
photon lifetime:

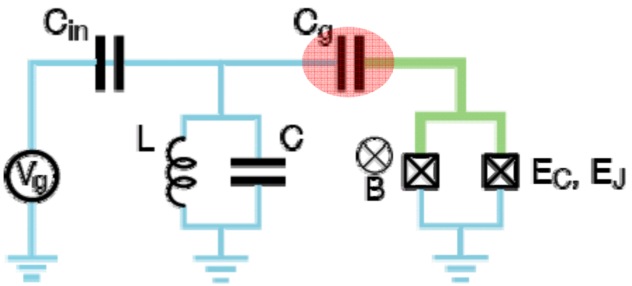
$$T_\kappa = 1/\kappa \approx 200\,\mathrm{ns}$$

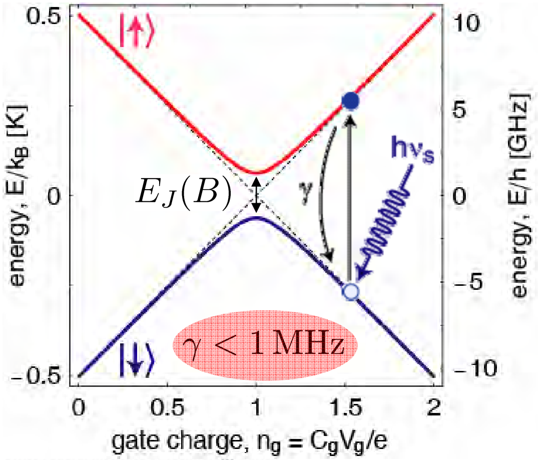
The Artificial Atom: A Cooper Pair Box



The Cooper Pair Box







coupling strength:

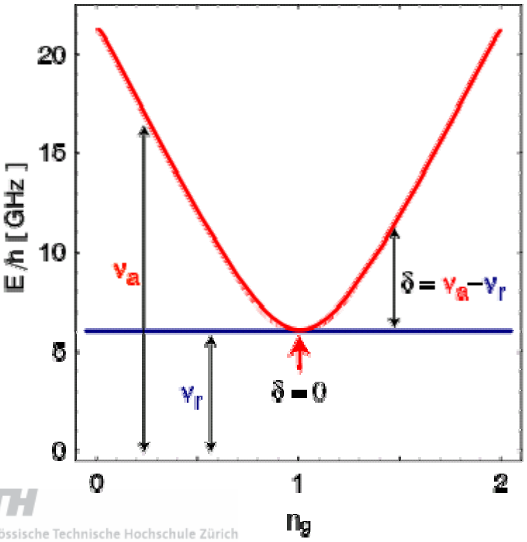
$$\hbar g = eV_{0,\mathrm{rms}}\frac{C_g}{C_\Sigma}$$

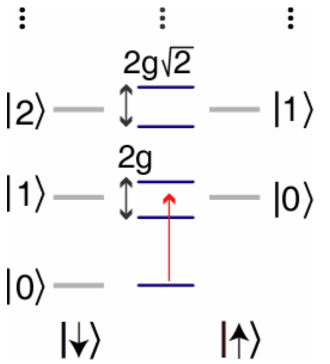
$$\Longrightarrow \nu_{\mathrm{vac}} = \frac{g}{\pi} \approx 1\ldots100\,\mathrm{MHz}$$

Resonant Strong Coupling Cavity QED

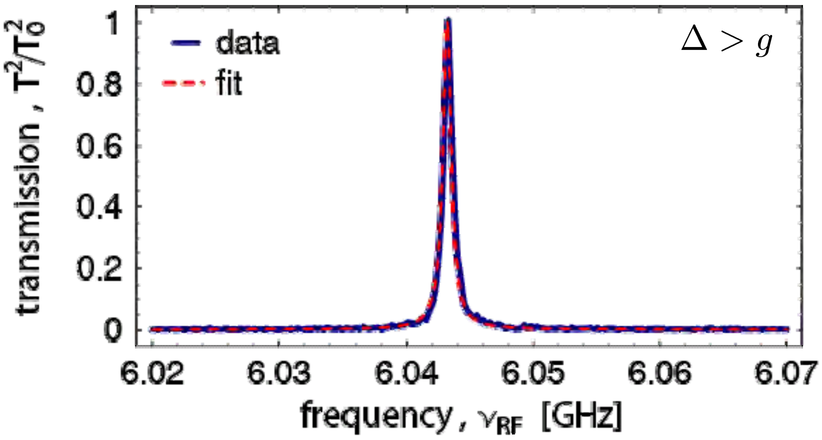
spectroscopic measurement of vacuum Rabi mode splitting

- tune into resonance ($\omega_a = \omega_r$)
- prepare ground state $|0,\downarrow\rangle$
- probe excited state spectrum with single photons





Bare Resonator Transmission Spectrum



- lorentzian line of width $\kappa/2\pi = 0.8\,\mathrm{MHz}$
- probed with intra-resonator photon number $n = 1$ at peak transmission

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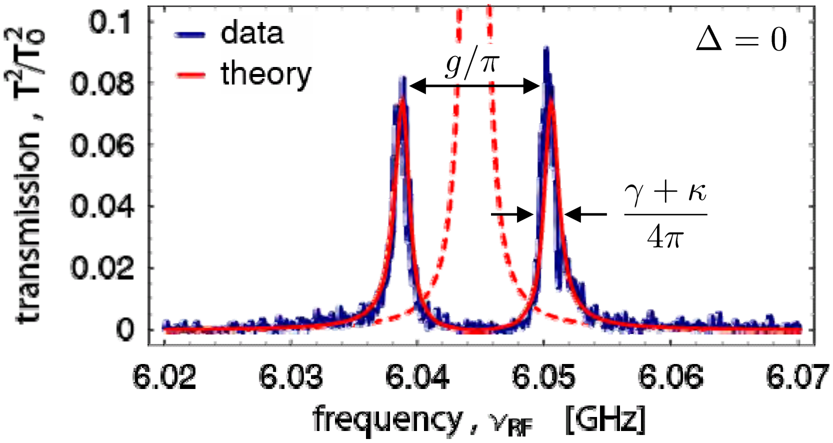
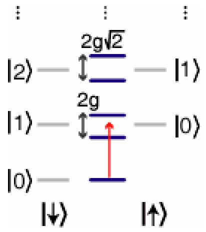
Vacuum Rabi Mode Splitting



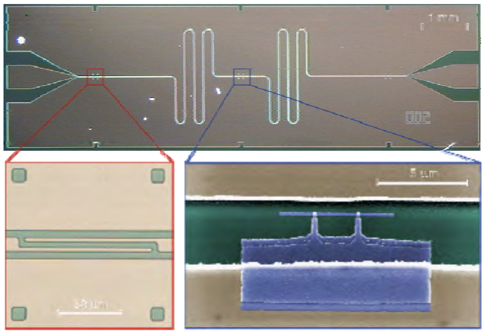


$$\begin{aligned} \nu_{\mathrm{vac}} &= 11.6\,\mathrm{MHz} \\ (g/2\pi &= 5.8\,\mathrm{MHz}) \\ \kappa/2\pi &= 0.8\,\mathrm{MHz} \\ \gamma/2\pi &= 0.7\,\mathrm{MHz} \end{aligned}$$

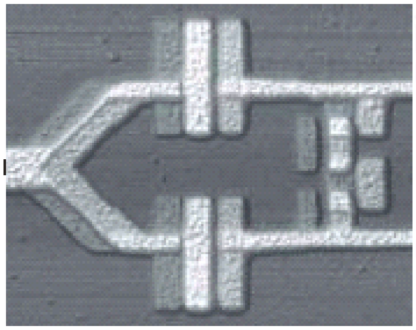
strong coupling $g \gg [\kappa, \gamma]$!

A. Wallraff, D. Schuster, ..., S. Girvin, and R. J. Schoelkopf,
Nature (London) **431**, 162 (2004)

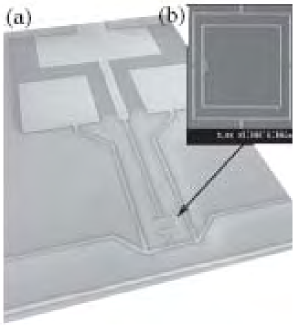
Coherent Dynamics with Single Photons



Yale University
charge qubit & transmission line res.
Nature (London) **431**, 162 (2004)



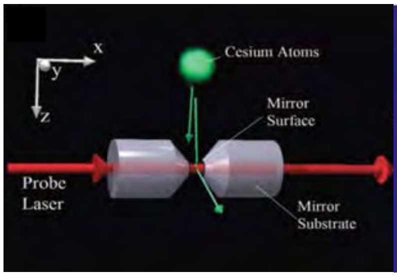
TU Delft
flux qubit & SQUID osc.
Nature (London) **431**, 159 (2004)



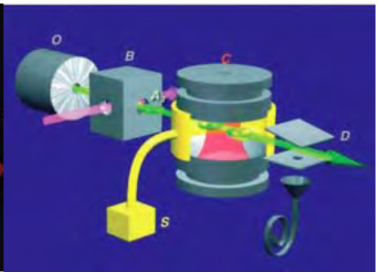
NTT
flux qubit & LC res.
PRL **96**, 127006 (2006)

circuit quantum electrodynamics (QED)

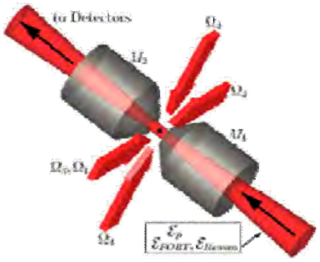
Strong Coupling Cavity QED



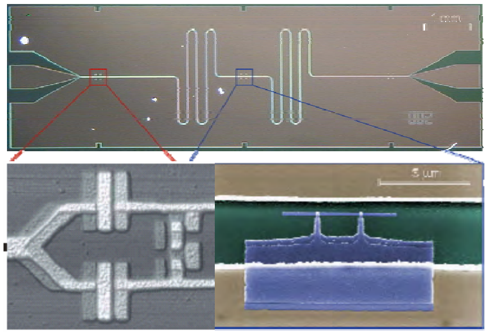
alkali atoms
Rempe, Kimble, ...



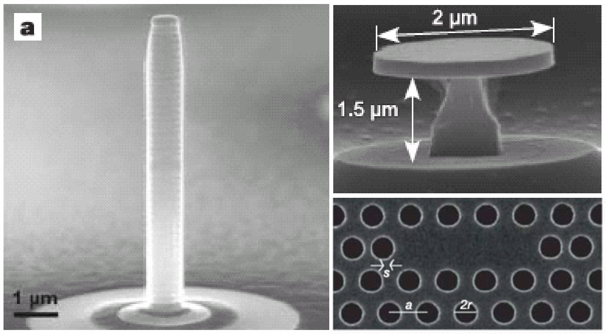
Rydberg atoms
Haroche, Walther, ...



single trapped atom
PRL **93**, 233603 (Dec. 2004)



superconductor flux and charge qubits
Nature (London) **431**, 159 & 162 (Sept. 2004)



semiconductor quantum dots
Nature (London) **432**, 197 & 200 (Nov. 2004)

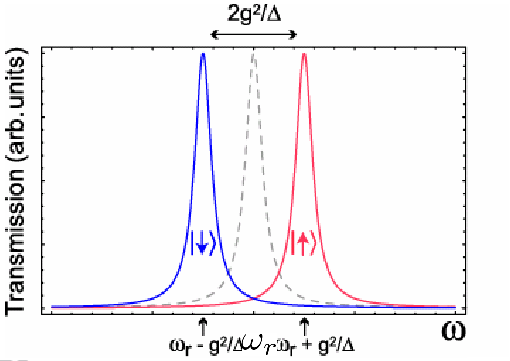
Dispersive (Non-Resonant) Qubit Readout

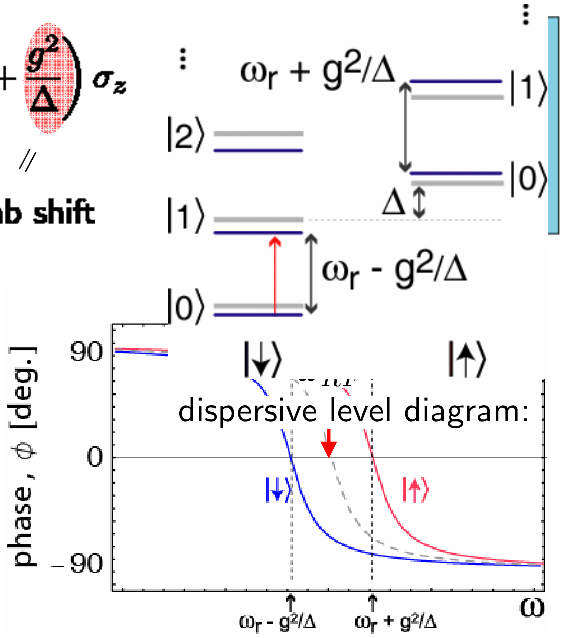
approximate diagonalization for $|\Delta| = |\omega_a - \omega_r| \gg g$

$$H \approx \hbar\left(\omega_r + \frac{g^2}{\Delta}\sigma_z\right)a^\dagger a + \frac{1}{2}\hbar\left(\omega_a + \frac{g^2}{\Delta}\right)\sigma_z$$

// cavity frequency shift *and* qubit ac-Stark shift

// Lamb shift

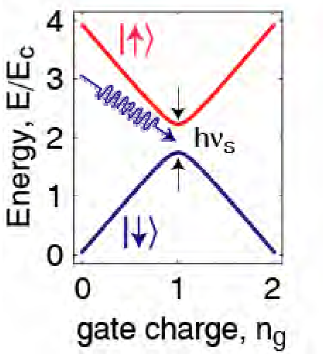


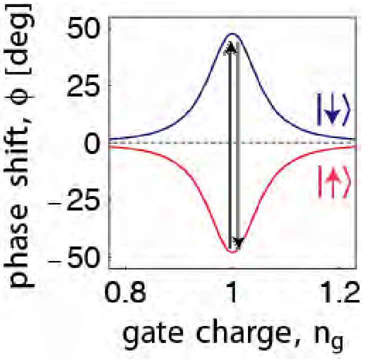


dispersive level diagram:

A. Blais *et al.*, *PRA* **69,** 062320 (2004)

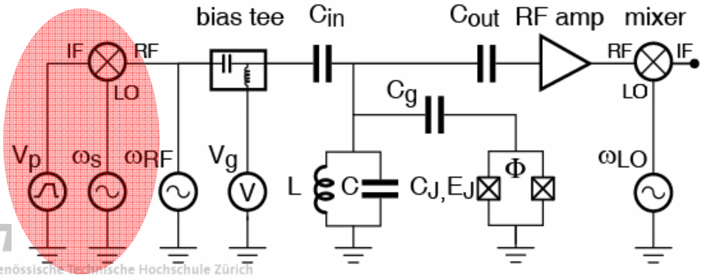
Coherent Control and Read-out in a Cavity





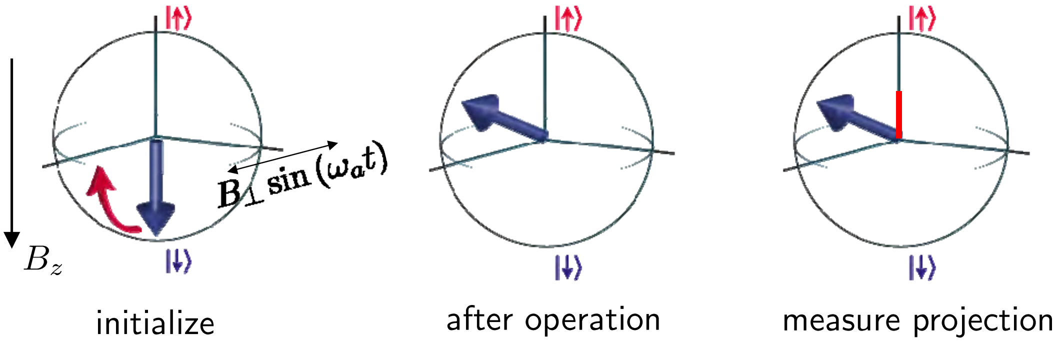
- apply resonant microwave pulse to qubit
- detect change of phase

realization:



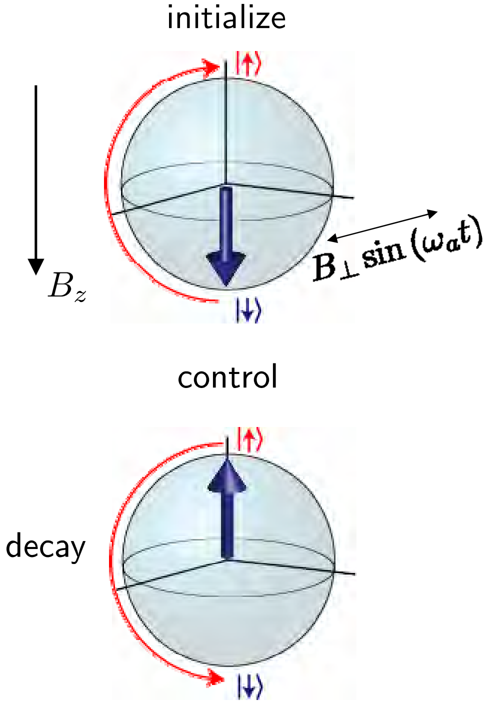
- simultaneous control and measurement

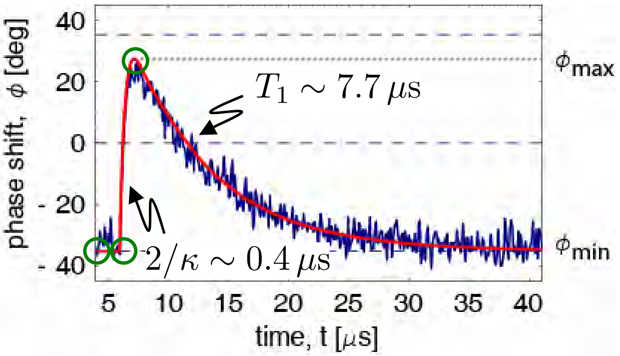
Coherent Control of a Qubit in a Cavity



- qubit state represented on a Bloch sphere
- NMR style operations
- vary length, amplitude and phase of pulse to control qubit state
- observe Rabi oscillations

Time-Resolved Dispersive QND Readout





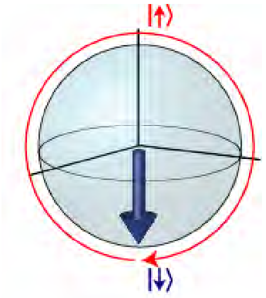
measurement properties:

- **continuous**
- **dispersive**
- **quantum non-demolition**
- **in good agreement with predictions**

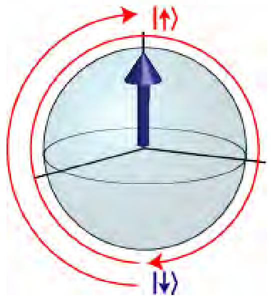
Wallraff, Schuster, Blais, ... Girvin, and Schoelkopf,
Phys. Rev. Lett. **95**, 060501 (2005)

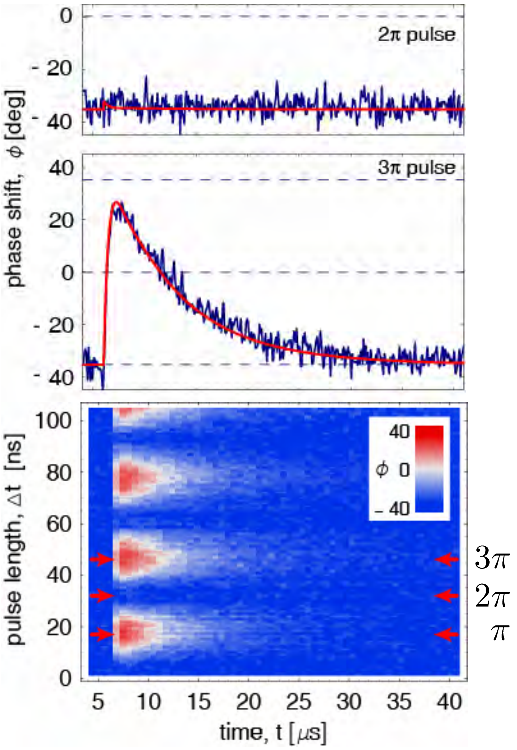
Varying the Control Pulse Length

2π rotation



3π rotation



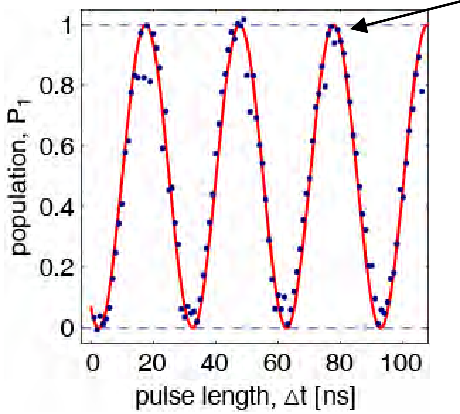


Wallraff, Schuster, Blais, ... Girvin, and Schoelkopf, *Phys. Rev. Lett.* **95**, 060501 (2005)

High Visibility Rabi Oscillations

Rabi oscillations:

visibility $95 \pm 5\%$



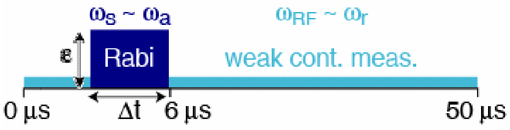
for superconducting qubits:

- first high visibility
- well characterized and understood measurement
- good control accuracy

A. Wallraff, D. I. Schuster, A. Blais, L. Frunzio, J. Majer,
S. M. Girvin, and R. J. Schoelkopf, *Phys. Rev. Lett.* **95**, 060501 (2005)

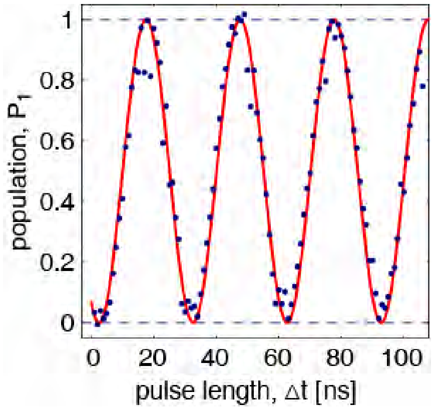
High Fidelity Control & Read Out

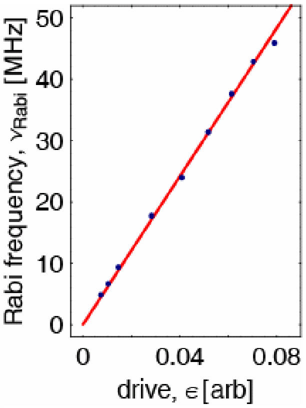
pulse scheme:



Rabi oscillations:

Rabi frequency:

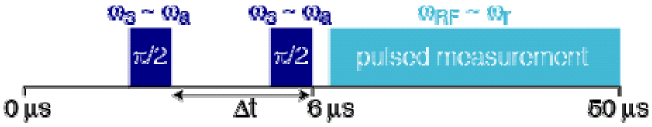


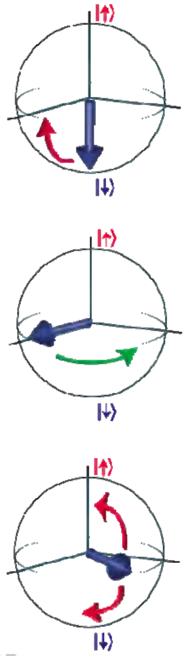


- high visibility $95 \pm 5\%$
- detailed understanding of qubit/read-out interaction

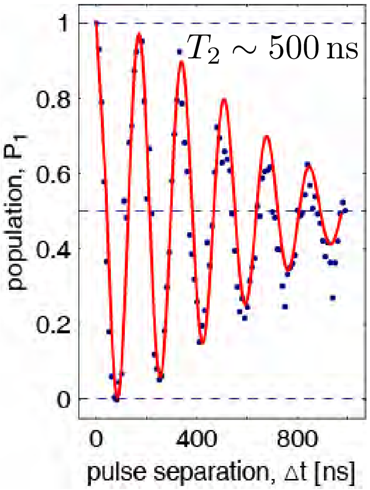
Long Coherence Time

pulse scheme:



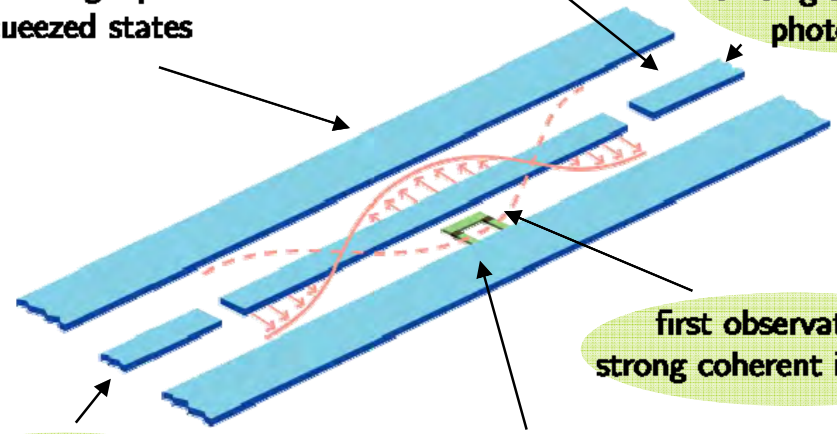


Ramsey fringes:



A. Wallraff *et al.*, *Phys. Rev. Lett.* **95**, 060501 (2005)

Circuit QED and Quantum Optics



in the future:
generate single photons
squeezed states

photon statistics:
coherent and
thermal states

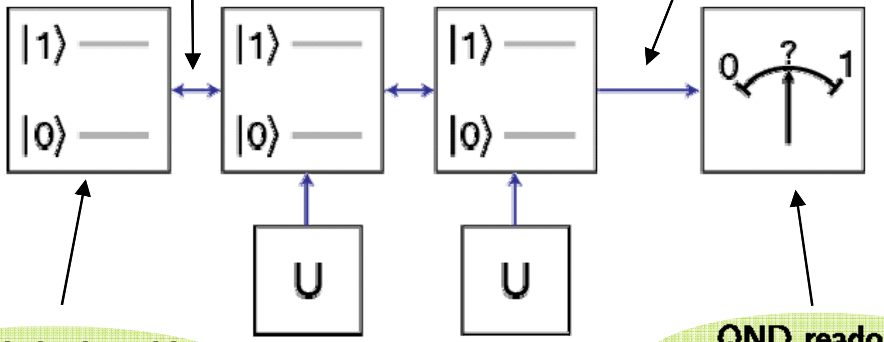
measurements in solid
on single microwave
photon level

first observation of
strong coherent interaction

realized cavities
with large vacuum fields
long photon lifetimes

artificial atom
with large dipole moment
long coherence time

Circuit QED and Quantum Computation



next steps:
use cavity as non-local inter-qubit coupling bus
demonstrate 2 bit gate and quantum algorithm

measurement process
and backaction
understood

realized single qubits
with long coherence times

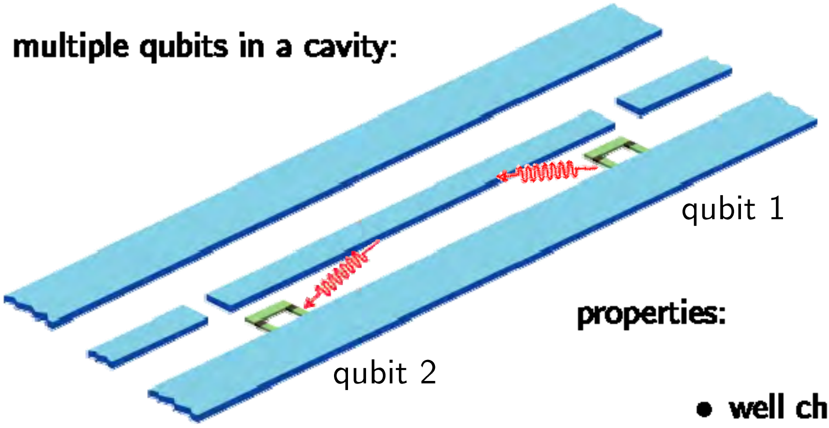
demonstrated
accurate control

QND readout
with high visibility

2 Qubits

Coupled Qubits

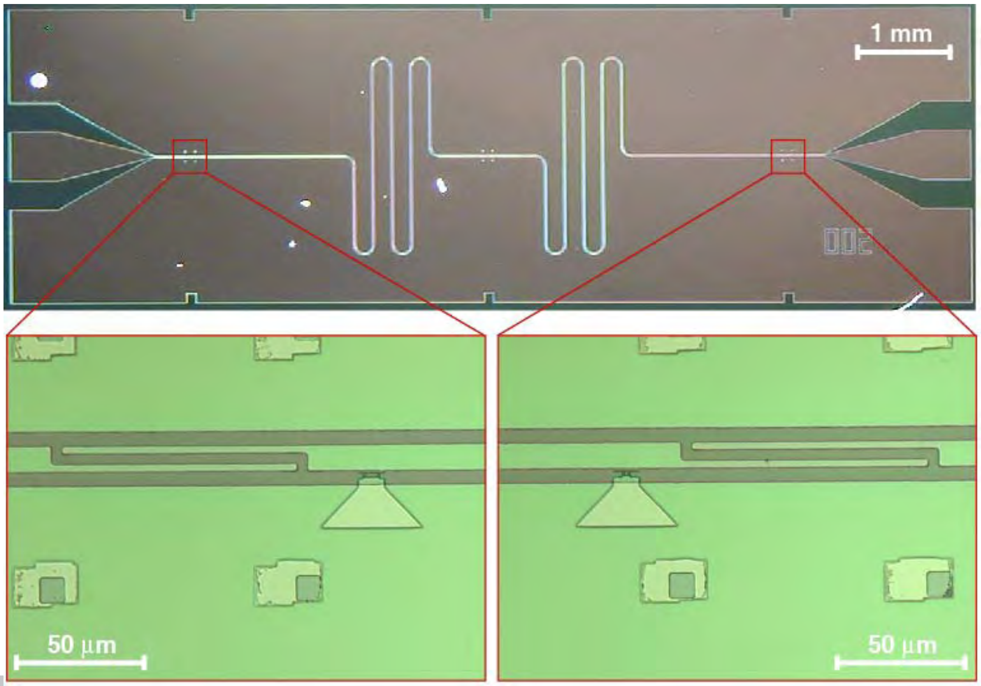
multiple qubits in a cavity:



properties:

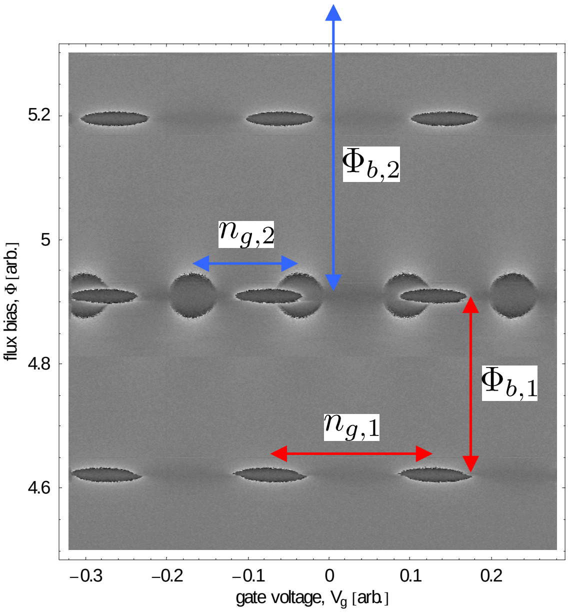
- **well characterized interaction**
- **controllable**
- **non-local**
- **scalable**

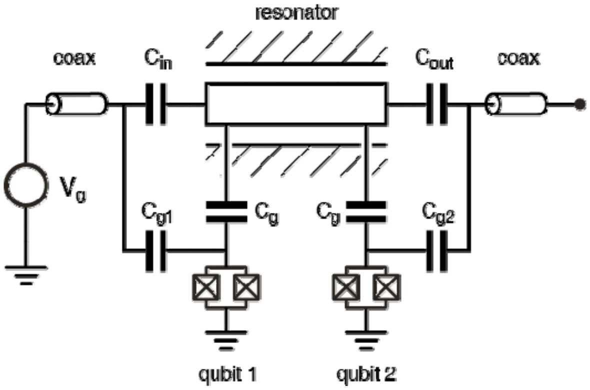
Coupled Qubit Device

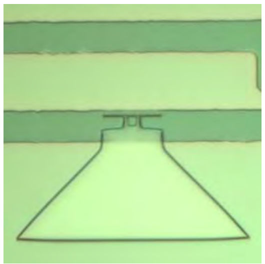


Resonator Response

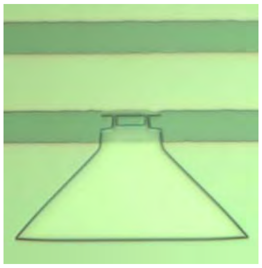
dispersive phase shift:







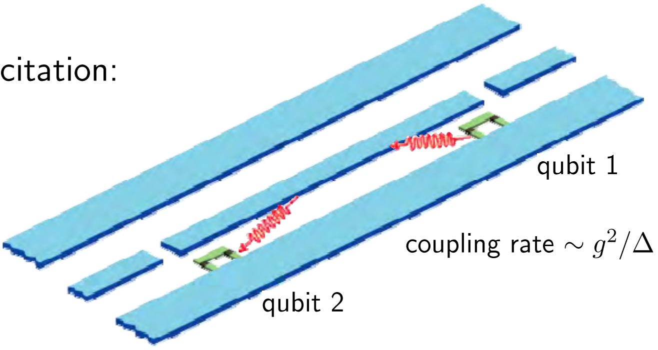
qubit 1

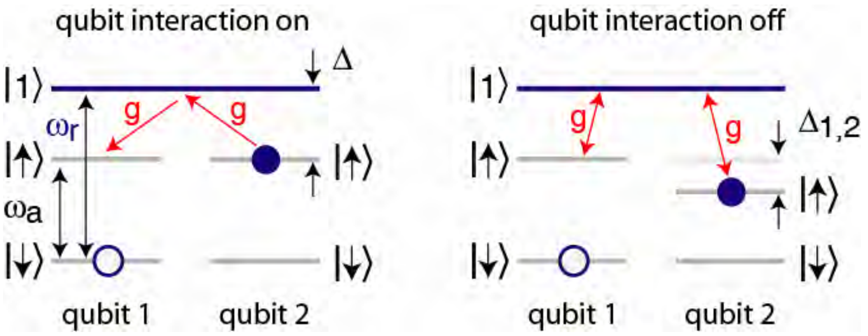


qubit 2

Coupling Scheme: Virtual Photon Exchange

coupling through virtual excitation:





truth table

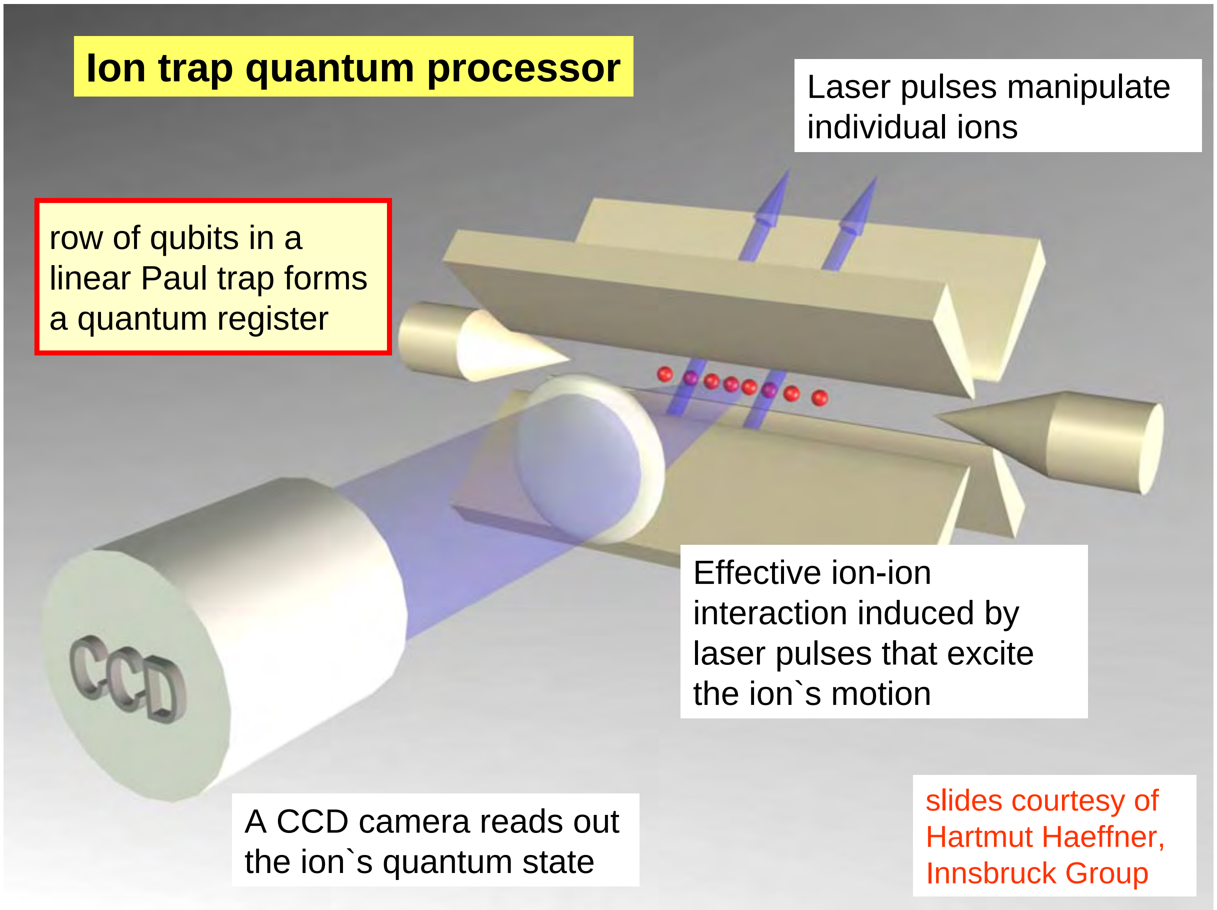
$|\uparrow\uparrow\rangle \rightarrow |\uparrow\uparrow\rangle$

$|\downarrow\downarrow\rangle \rightarrow |\downarrow\downarrow\rangle$

$|\downarrow\uparrow\rangle \rightarrow |\uparrow\downarrow\rangle$

$|\uparrow\downarrow\rangle \rightarrow |\downarrow\uparrow\rangle$

A. Blais, R.-S. Huang, A. Wallraff, S. M. Girvin, and R. J. Schoelkopf, *PRA* **69,** 062320 (2004)



Ion trap quantum processor

Laser pulses manipulate individual ions

row of qubits in a linear Paul trap forms a quantum register

Effective ion-ion interaction induced by laser pulses that excite the ion`s motion

A CCD camera reads out the ion`s quantum state

slides courtesy of Hartmut Haeffner, Innsbruck Group

Meeting the DiVincenzo criteria with trapped ions

criterion	physical implementation	
scalable qubits	internal atomic transitions (2-level-systems)	linear traps (trap arrays)
initialization	laser cooling, state preparation	optical pumping, laser pulses
long coherence times	narrow transitions (optical, microwave)	coherence time ~ ms - min
universal quantum gates	single qubit operations, two-qubit operations	Rabi oscillations Cirac-Zoller CNOT
qubit measurement	quantum jump detection	individual ion fluorescence
convert qubits to flying qubits	coupling of ions with high finesse cavity	CQED, bad cavity limit
faithfully transmit flying qubits	coupling of cavities via fiber (photonic channel)	coupling pulse sequences (CZKM)

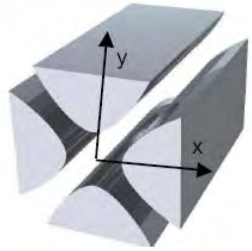
Trapping Individual Ions

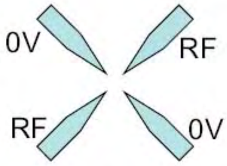
Linear Paul trap

2D rf-trap + static potential

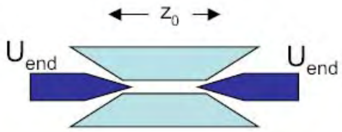
I. Waki et al., Phys. Rev. Lett. 68, 2007 (1992)
M.G. Raizen et al., Phys. Rev. A 45, 6493 (1992)

plug the ends of a mass filter by positive electrodes:



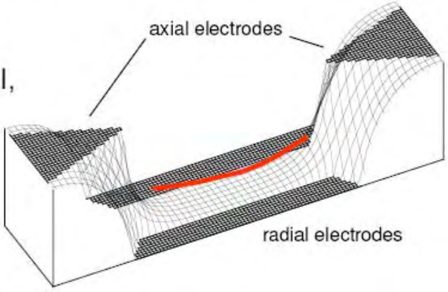


mass filter blade design

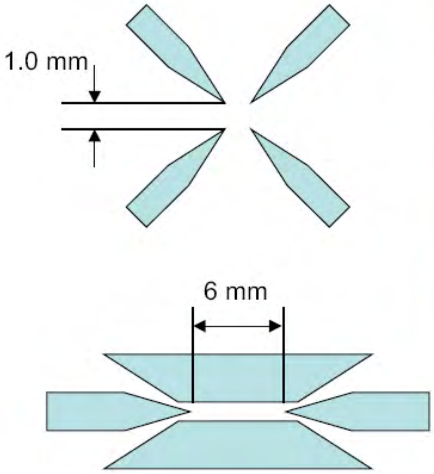


side view

numerically calculate the axial electric potential,
fit parabula into the potential
and get the axial trap frequency



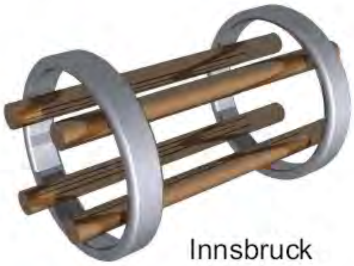
Innsbruck: Linear ion trap (2000)



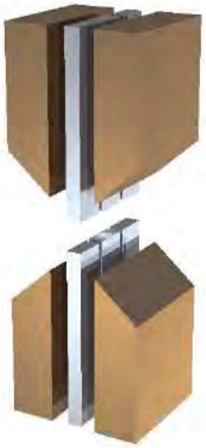
$$\omega_z \approx 0.7 - 2\,\text{MHz} \qquad \omega_{x,y} \approx 1.5 - 4\ \text{MHz}$$

Linear ion traps

Paul mass filter



Innsbruck
Ann Arbor



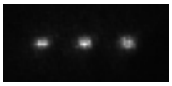
Munich

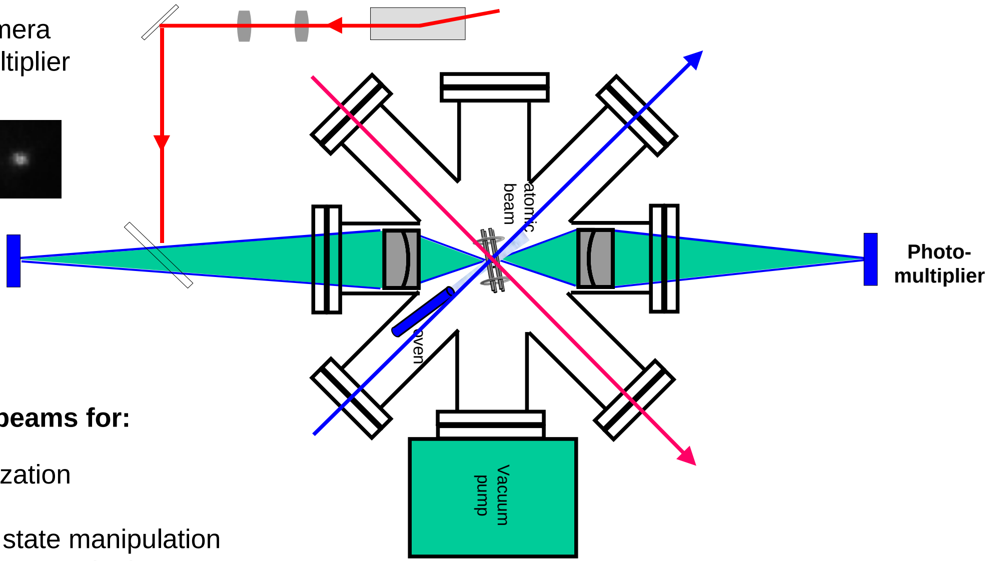
Boulder, Mainz, Aarhus

Experimental setup

Fluorescence detection by

CCD camera
photomultiplier

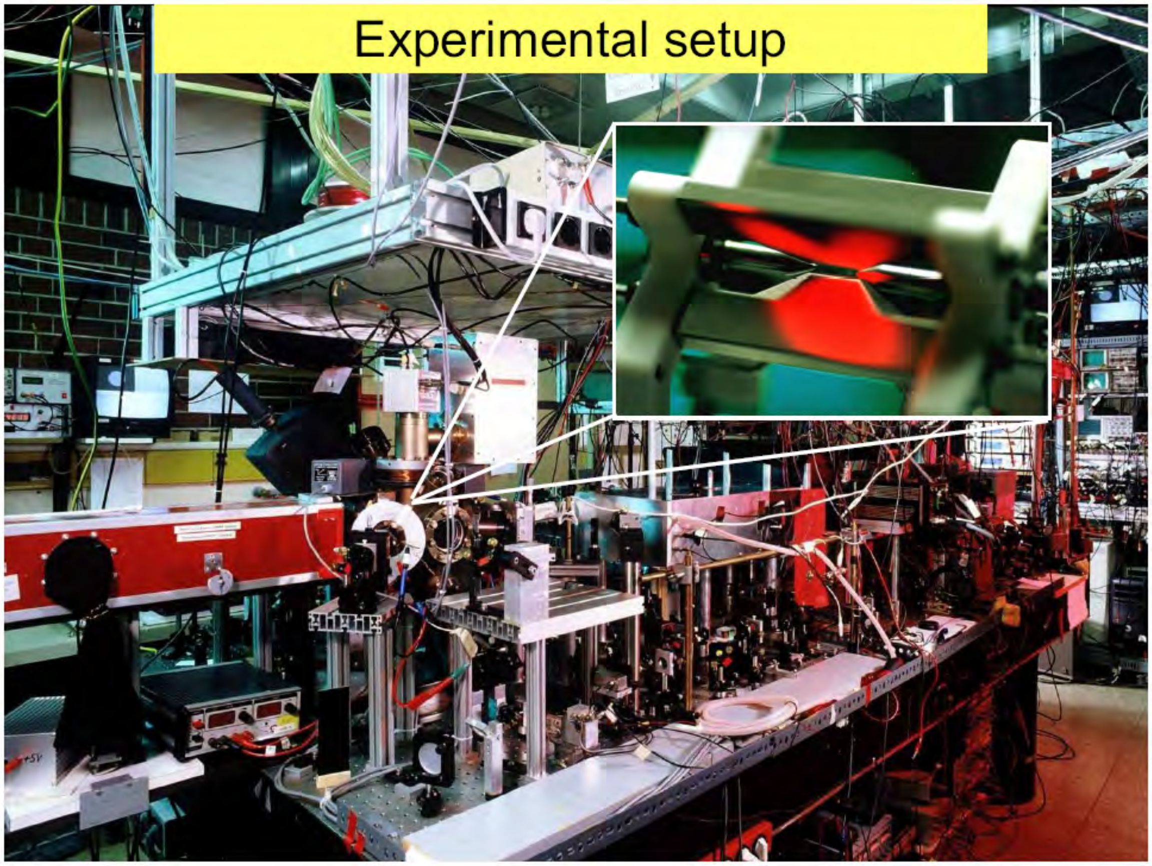




Laser beams for:

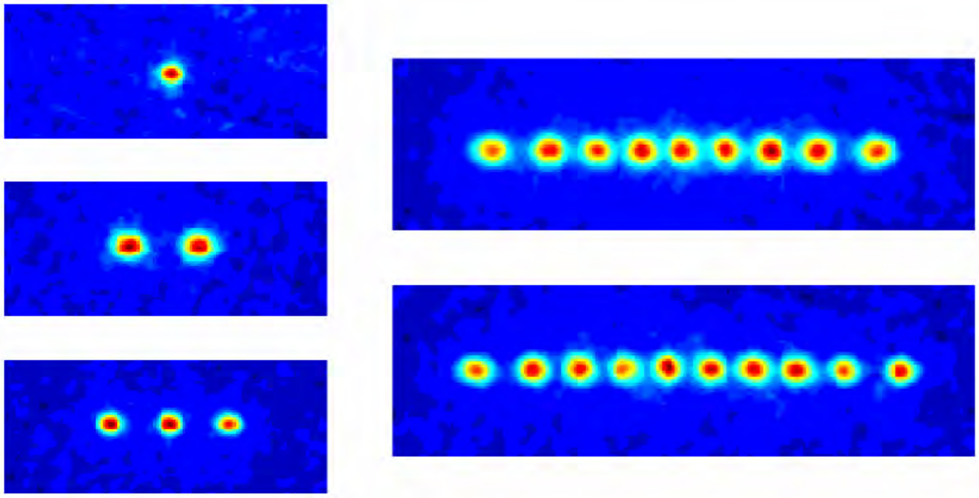
photoionization
cooling
quantum state manipulation
fluorescence excitation

Experimental setup



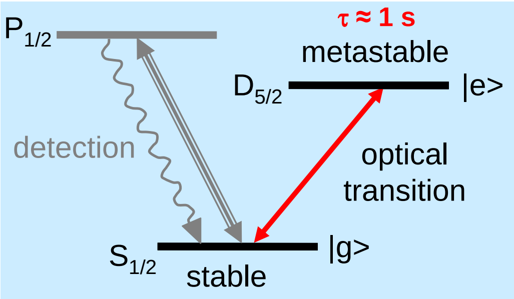
Ion strings

In strongly anisotropic traps: Formation of linear strings of ions



Ions as Quantum Bits

Ions with optical transition to metastable level: $^{40}Ca^+$,$^{88}Sr^+$,$^{172}Yb^+$

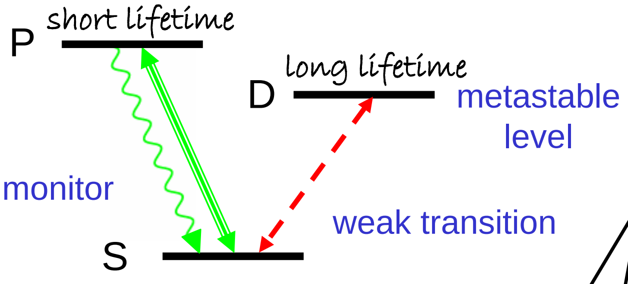


Qubit levels: $S_{1/2}$, $D_{5/2}$

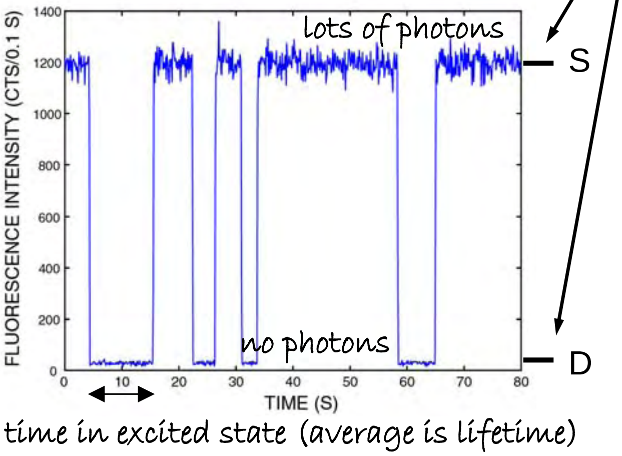
Qubit transition: Quadrupole transition $S_{1/2} – D_{5/2}$

Detection of Ion Quantum State

Quantum jumps: spectroscopy with quantized fluorescence



absorption and emission cause fluorescence steps (digital quantum jump signal)

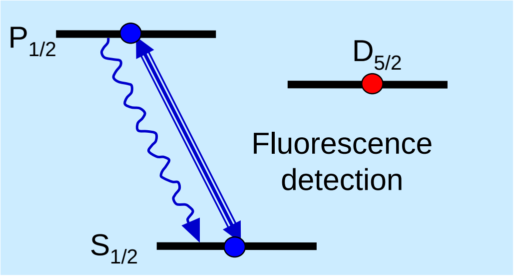


„ Quantum jump technique“
„ Electron shelving technique“

Observation of quantum jumps:

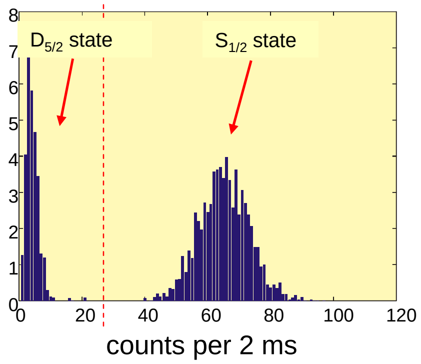
Nagourney et al., PRL **56**,2797 (1986),
Sauter et al., PRL **57**,1696 (1986),
Bergquist et al., PRL **57**,1699 (1986)

Electron shelving for quantum state detection



1. Initialization in a pure quantum state
2. Quantum state manipulation on $S_{1/2} – D_{5/2}$ transition
3. Quantum state measurement by fluorescence detection

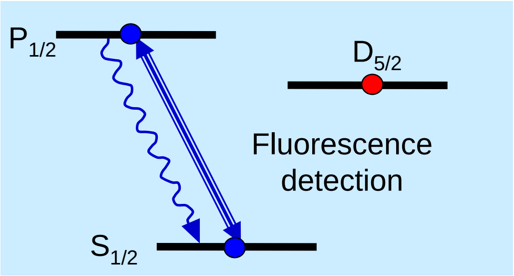
One ion : Fluorescence histogram



50 experiments / s

Repeat experiments 100-200 times

Electron shelving for quantum state detection



1. Initialization in a pure quantum state
2. Quantum state manipulation on $S_{1/2} – D_{5/2}$ transition
3. Quantum state measurement by fluorescence detection

Two ions:

Spatially resolved detection with CCD camera:

50 experiments / s

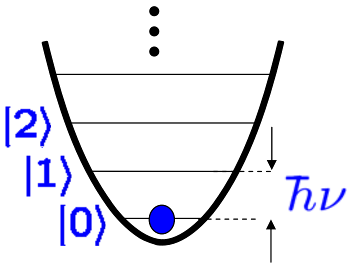
Repeat experiments 100-200 times

Mechanical Motion of Ions in their Trapping Potential:

Mechanical **Quantum harmonical oscillator**

Extension of the ground state:

harmonic trap



$$x = \sqrt{\frac{\hbar}{2m\nu}}(a + a^\dagger)$$

$$\langle 0|x^2|0\rangle = \frac{\hbar}{2m\nu}\langle 0|(a + a^\dagger)^2|0\rangle \quad = \frac{\hbar}{2m\nu}$$

$$\left.\begin{array}{l}\nu = (2\pi)1\,\mathrm{MHz} \\ \mathrm{m{=}40\ u}\end{array}\right\} \quad \langle x^2\rangle^{1/2} = \sqrt{\frac{\hbar}{2m\nu}} \approx 11\mathrm{nm}$$

Size of the wave packet << wavelength of visible light

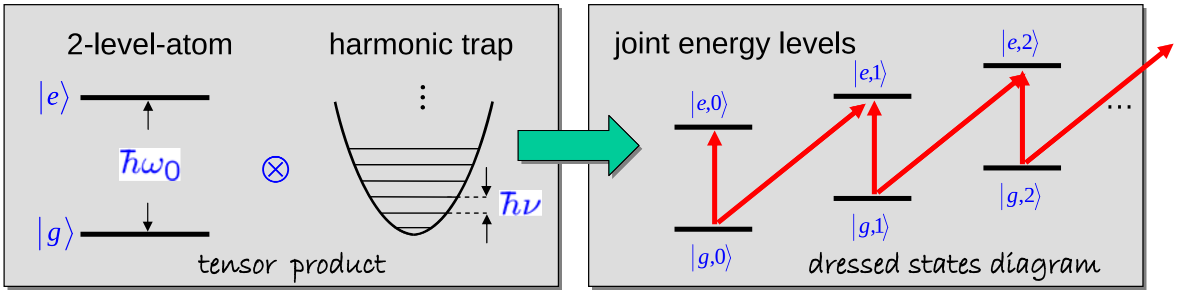
Energy scale of interest:

$$\hbar\nu = k_B T \quad \longrightarrow \quad T = \frac{\hbar\nu}{k_B} \approx 50\mu\mathrm{K}$$

ions need to be very cold to be in their vibrational ground state

---

An Ion Coupled to a Harmonic Oscillator



$H^{(el)}$ $H^{(m)}$

Laser – ion interactions

Approximations:

Ion: Electronic structure of the ion approximated by two-level system

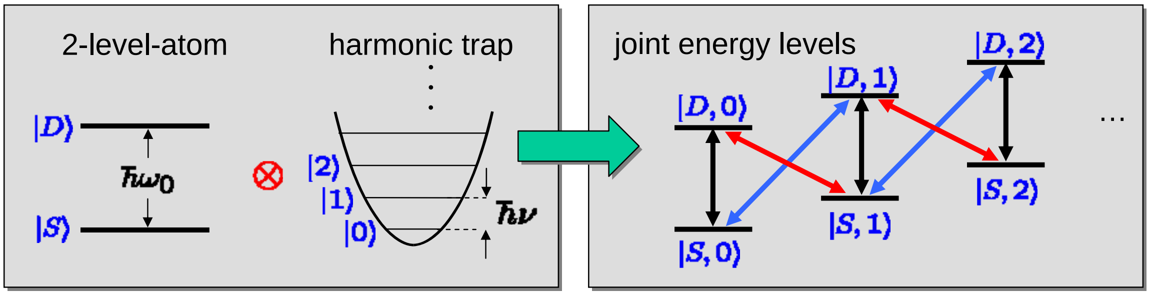
(laser is (near-) resonant and couples only two levels)

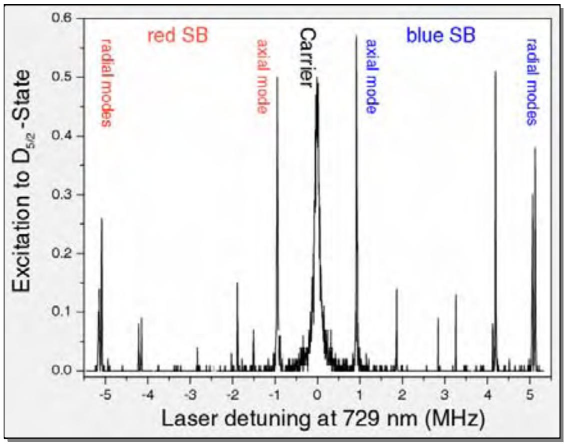
$$H^{(el)} = \hbar\frac{\omega_0}{2}(|e\rangle\langle e| - |g\rangle\langle g|)$$

Trap: Only a single harmonic oscillator taken into account

$$H^{(m)} = \hbar\nu a^\dagger a$$

External degree of freedom: ion motion



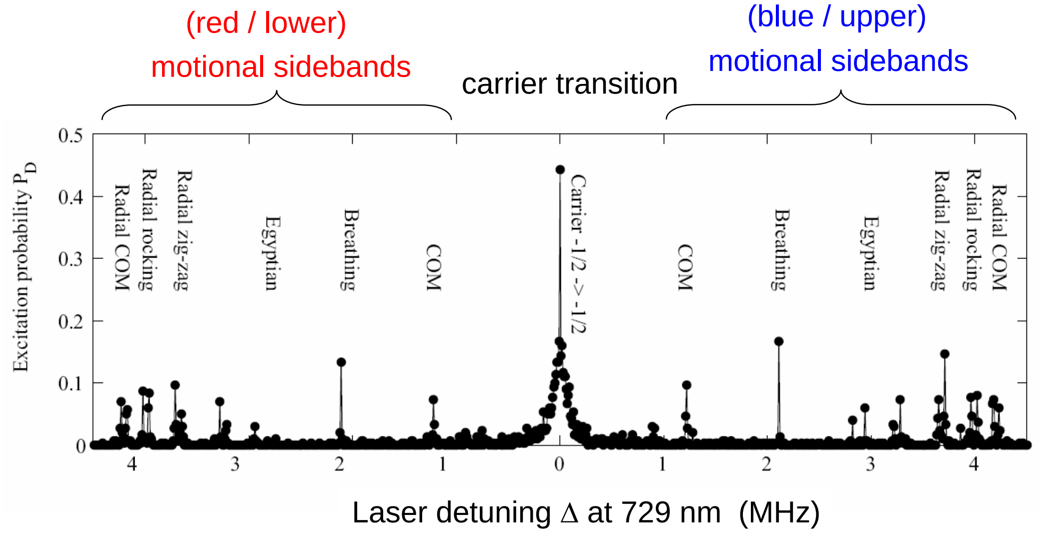


ion transition
frequency 400 THz

trap frequency 1 MHz

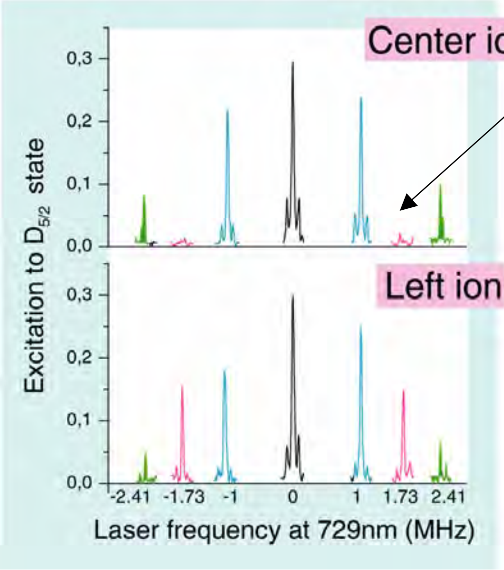
A closer look at the excitation spectrum (3 ions)

$$S_{1/2}, m = -1/2 \longleftrightarrow D_{5/2}, m = -1/2$$



many different vibrational modes of ions in the trap

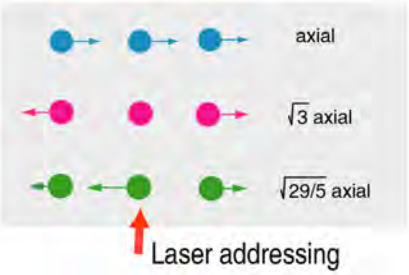
Sideband spectra of individually addressed three ions



breathing mode cannot be excited on center ion

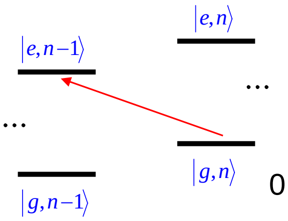
Eigen-vectors and Eigen-values

V1=	{ 0.577	0.577	0.577 }	1
V2=	{ -0.707	0	0.707 }	1.73
V3=	{ -0.408	0.817	-0.408 }	2.41



red and blue side bands can be observed because vibrational motion of ions is not cooled (in this example)

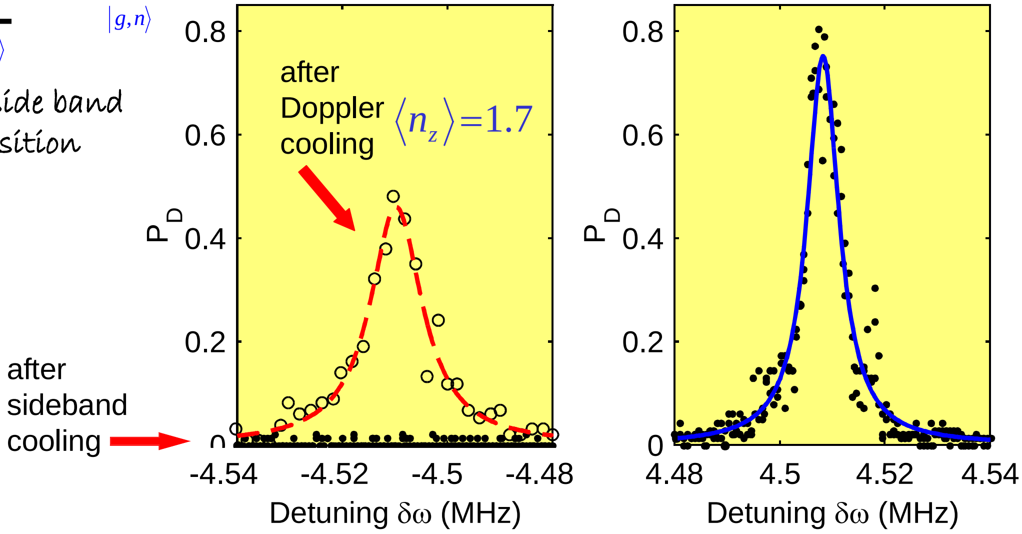
Cooling of the vibrational modes



red side band transition

Sideband absorption spectra

99.9 % ground state population



red sideband

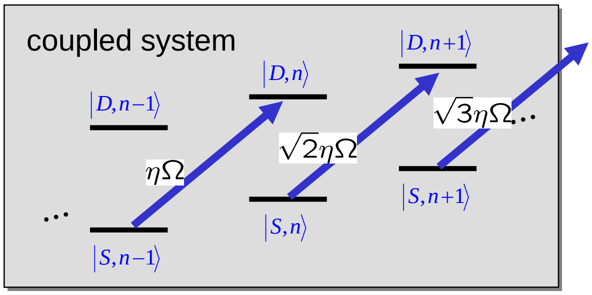
blue sideband

But also controlled excitation of the vibrational modes

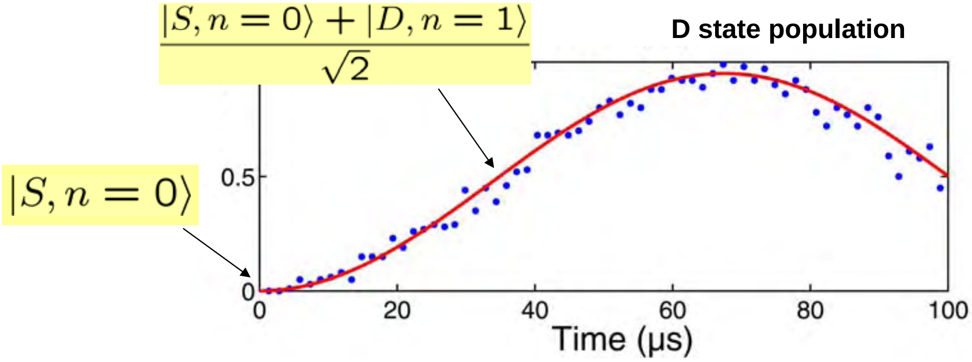
Coherent excitation on the sideband

„Blue sideband“ pulses:

$$|S\rangle|n\rangle \longleftrightarrow |D\rangle|n+1\rangle$$



$\theta = \pi/2$: Entanglement between internal and motional state !



Single qubit operations

Arbitrary qubit rotations:

z-rotations

- Laser slightly detuned from carrier resonance

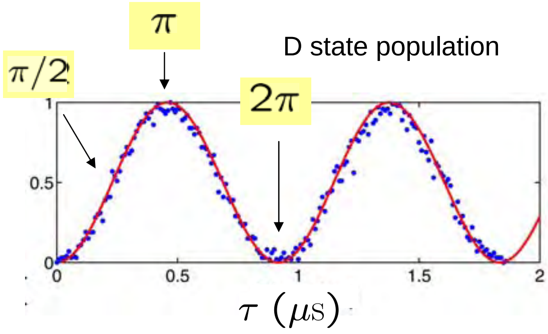
(z-rotations by off-resonant laser beam creating ac-Stark shifts)

or:

- Concatenation of two pulses with rotation axis in equatorial plane

x,y-rotations

Gate time : 1-10 µs ⟷ Coherence time : 2-3 ms

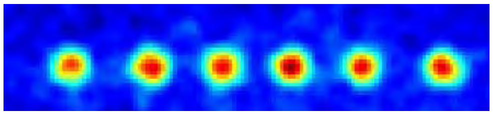


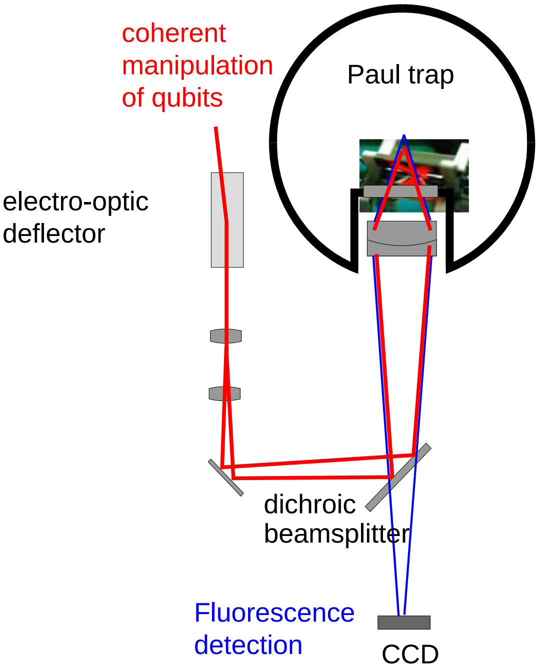
limited by

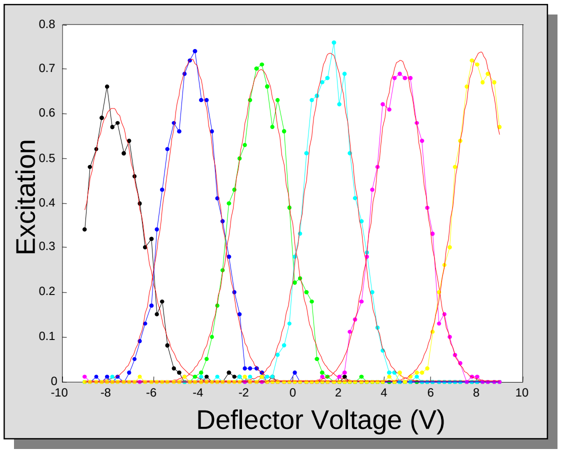
- magnetic field fluctuations
- laser frequency fluctuations

(laser linewidth δν<100 Hz)

Addressing the qubits





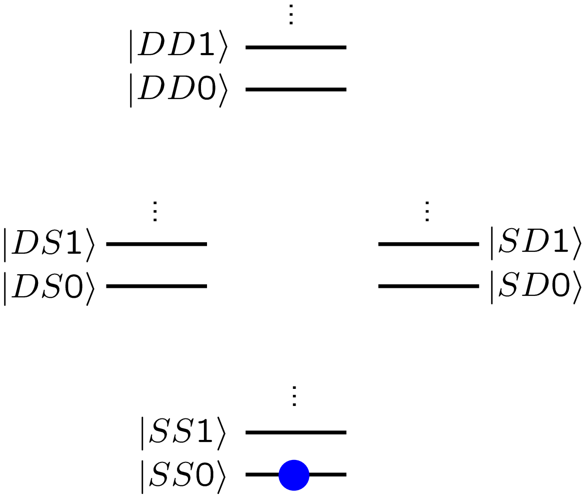


- inter ion distance: ~ 4 µm
- addressing waist: ~ 2 µm

< 0.1% intensity on neighbouring ions

Generation of Bell states

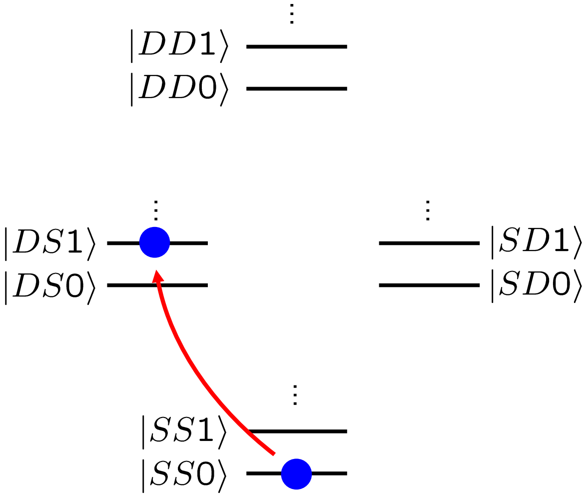
generation of entanglement between two ions



Pulse sequence:

$|SS0\rangle$

Generation of Bell states



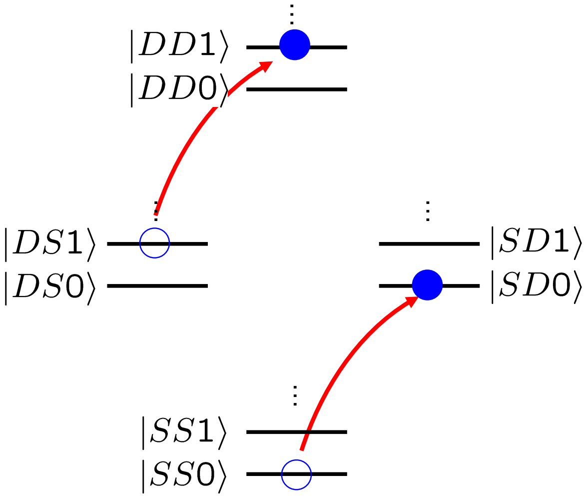
Pulse sequence:

Ion 1: π/2 , blue sideband

creates entangled state
between qubit 1 and oscillator

$$|SS0\rangle + |DS1\rangle$$

Generation of Bell states



Pulse sequence:

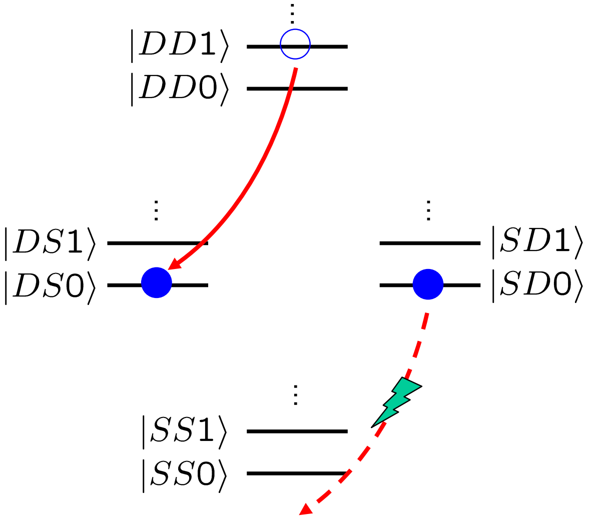
Ion 1: π/2 , blue sideband

Ion 2: π , carrier

excites qubit 2

$$|SD0\rangle + |DD1\rangle$$

Generation of Bell states



|SD0> is non-resonant and remains unaffected

Pulse sequence:

Ion 1: π/2 , blue sideband

Ion 2: π , carrier

Ion 2: π , blue sideband

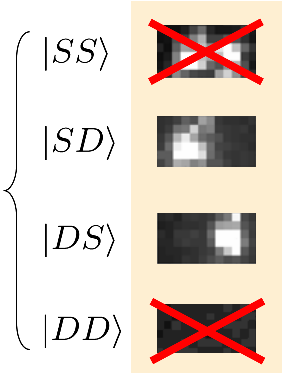
takes qubit 2 (with one oscillator excitation) back to ground state and removes excitation from oscillator

$$(|SD\rangle + |DS\rangle)|0\rangle$$

Bell state analysis

$$|SD\rangle + |DS\rangle$$

Fluorescence detection with CCD camera:

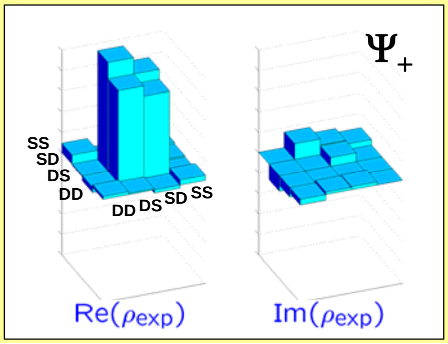


Coherent superposition or incoherent mixture ?

What is the relative phase of the superposition ?

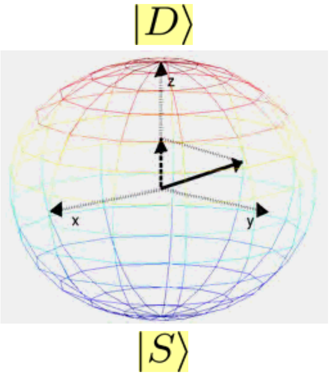
tomography of qubit states (= full measurement of x,y,z components of both qubits and its correlations)

→ Measurement of the density matrix:



Obtaining a single qubits density matrix

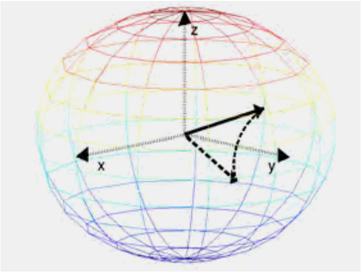
(a naïve persons point of view)



A measurement yields the *z*-component of the Bloch vector

=> Diagonal of the density matrix

$$\rho = \begin{pmatrix} P_S & C - iD \\ C + iD & P_D \end{pmatrix}$$

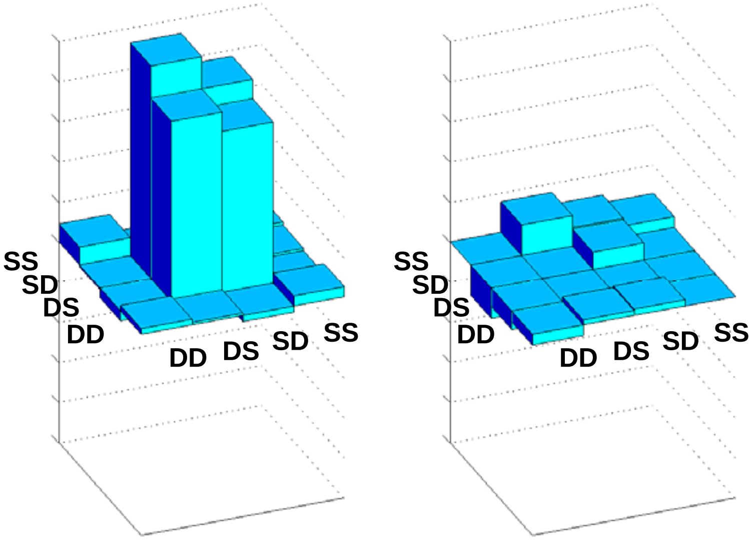


Rotation around the *x*- or the *y*-axis prior to the measurement yields the phase information of the qubit.

=> coherences of the density matrix

as discussed before!

Bell state reconstruction



$\mathrm{Re}(\rho_{\mathrm{exp}})$ $\mathrm{Im}(\rho_{\mathrm{exp}})$

$|SD\rangle + |DS\rangle$

F=0.91

Phase gate ⇔ CNOT

$$\begin{array}{ccccccc} & \xrightarrow{R_1^C(\frac{\pi}{2},\frac{\pi}{2})} & & \xrightarrow{\text{Phasegate}} & & \xrightarrow{R_1^C(\frac{\pi}{2},-\frac{\pi}{2})} & \\ \hline |0\rangle\otimes|0\rangle & & |0\rangle\otimes(|0\rangle+|1\rangle) & & |0\rangle\otimes(|0\rangle+|1\rangle) & & |0\rangle\otimes|0\rangle \\ |0\rangle\otimes|1\rangle & & |0\rangle\otimes(|0\rangle-|1\rangle) & & |0\rangle\otimes(|0\rangle-|1\rangle) & & |0\rangle\otimes|1\rangle \\ |1\rangle\otimes|0\rangle & & |1\rangle\otimes(|0\rangle+|1\rangle) & & |1\rangle\otimes(|0\rangle-|1\rangle) & & |1\rangle\otimes|1\rangle \\ |1\rangle\otimes|1\rangle & & |1\rangle\otimes(|0\rangle-|1\rangle) & & |1\rangle\otimes(|0\rangle+|1\rangle) & & |1\rangle\otimes|0\rangle \end{array}$$

Both, the phase gate as well the CNOT gate can be converted into each other with single qubit operations.

$$R^C(\pi/2,\pi/2) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

Together with the three single qubit gates,
we can implement any unitary operation!

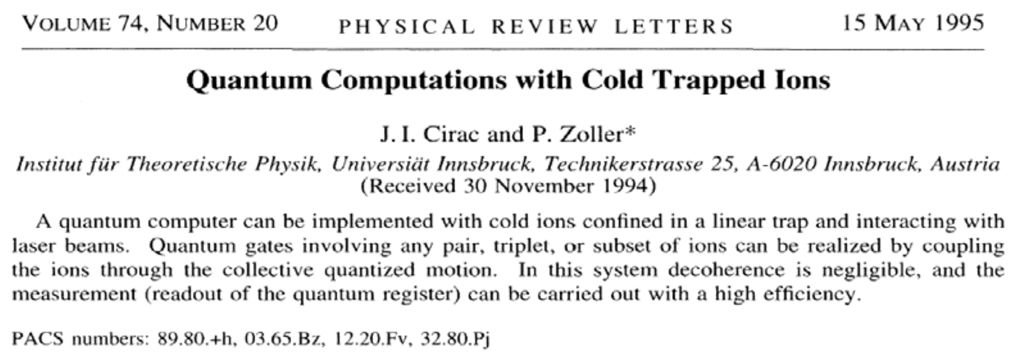
$$R^C(\pi/2,-\pi/2) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$00 \to 00$
$01 \to 01$
$10 \to 10$
$11 \to -11$

phase gate

$R^C(\theta,\varphi)$

Quantum gate proposals with trapped ions



Volume 74, Number 20 PHYSICAL REVIEW LETTERS 15 May 1995

Quantum Computations with Cold Trapped Ions

J. I. Cirac and P. Zoller*
Institut für Theoretische Physik, Universiät Innsbruck, Technikerstrasse 25, A-6020 Innsbruck, Austria

(Received 30 November 1994)

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PACS numbers: 89.80.+h, 03.65.Bz, 12.20.Fv, 32.80.Pj

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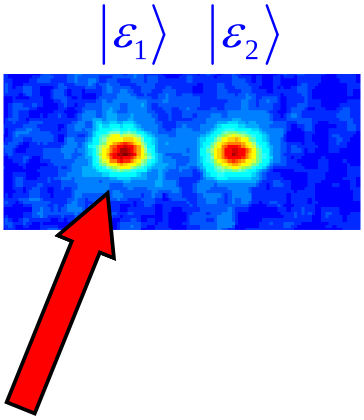
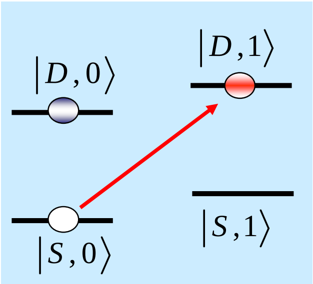
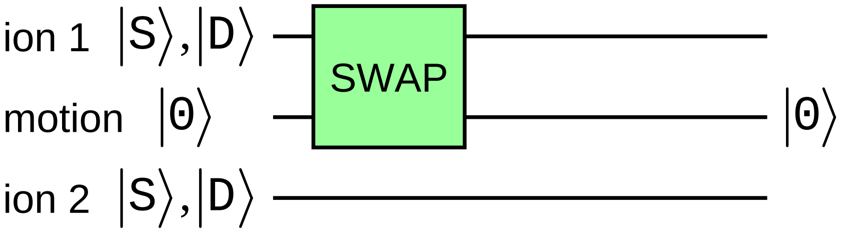
- Geometric phases
- Leibfried & Wineland

Cirac - Zoller two-ion phase gate







ion 1 $|S\rangle,|D\rangle$

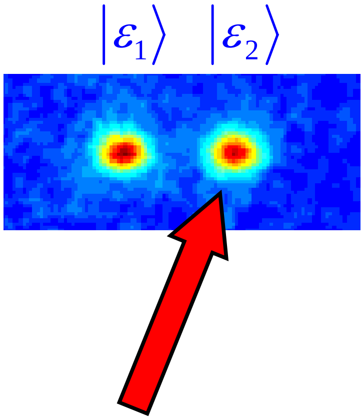
motion $|0\rangle$

ion 2 $|S\rangle,|D\rangle$

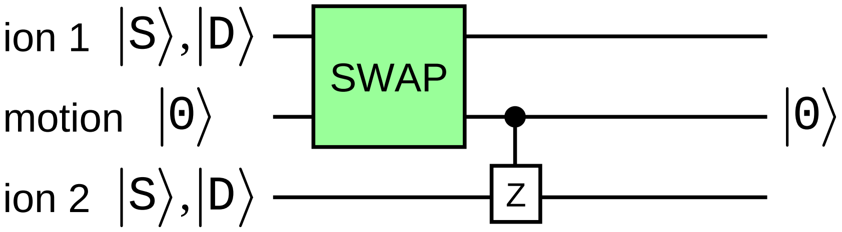
SWAP

$|0\rangle$

Cirac - Zoller two-ion phase gate



Phase gate using
the motion and
the target bit.



ion 1 $|S\rangle,|D\rangle$

motion $|0\rangle$

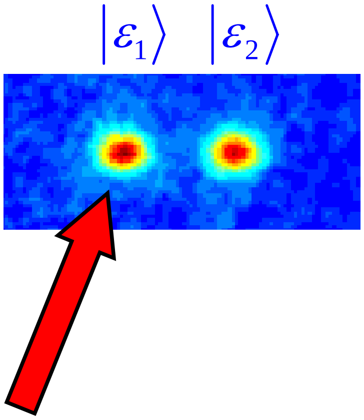
ion 2 $|S\rangle,|D\rangle$

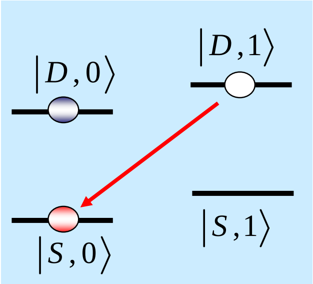
SWAP

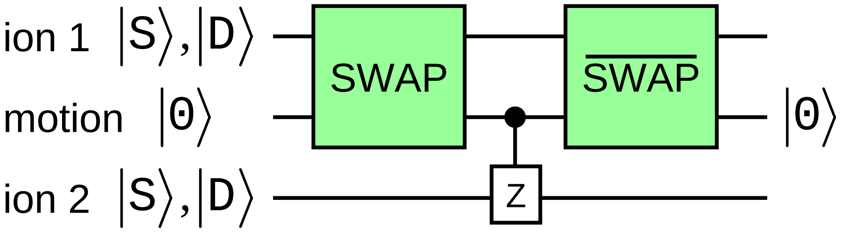
Z

$|0\rangle$

Cirac - Zoller two-ion phase gate







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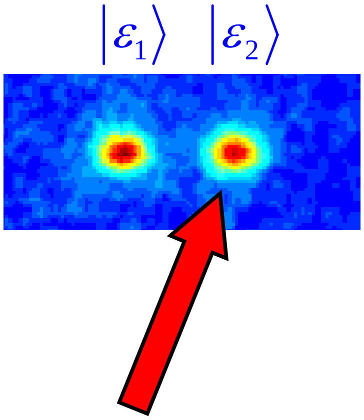
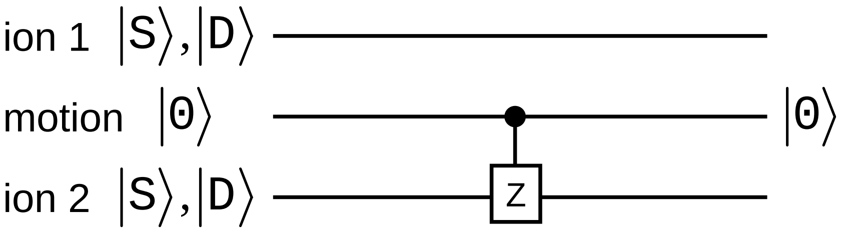
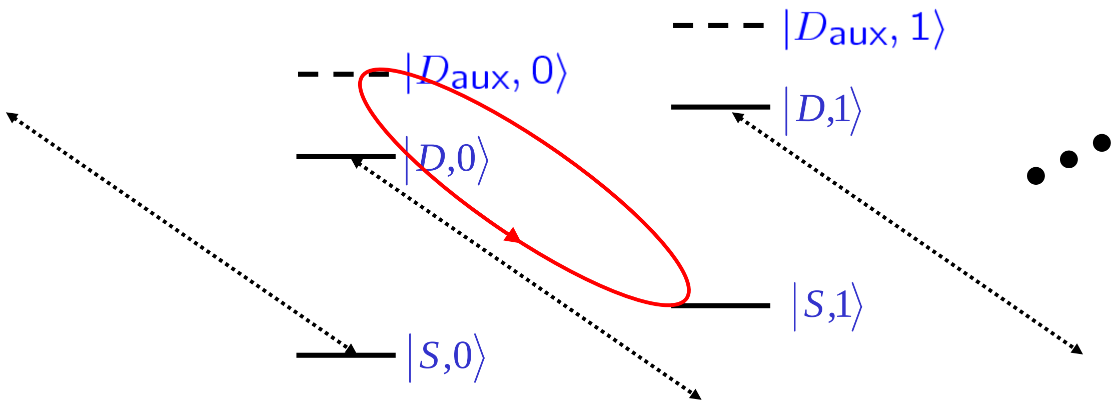


Phase gate using
the motion and
the target bit.



Cirac-Zoller phase gate: the key step



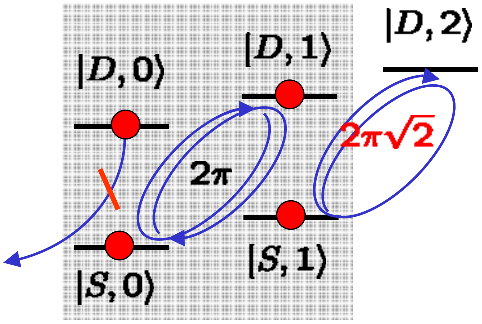
A 2π pulse is applied to only one of ion-(crystal)s states => only one states acquires a phase factor of –1.

An additional Zeeman level can be used as the auxilary state.

=> gate is sensitive to magnetic field fluctuations!

How do you do with just a two-level system?

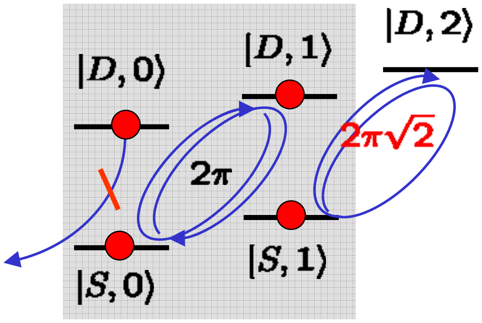
$$\begin{array}{ccccc} & |D,0\rangle & |S,0\rangle & |D,1\rangle & |S,1\rangle \\ & 1 & 0 & 0 & 0 \\ U_\Phi = & 0 & -1 & 0 & 0 \\ & 0 & 0 & -1 & 0 \\ & 0 & 0 & 0 & \boxed{?} \end{array}$$



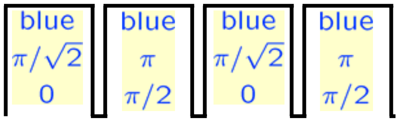
Phase gate

$$U_\Phi = \begin{array}{c} \\ \end{array}\begin{array}{cccc} |D,0\rangle & |S,0\rangle & |D,1\rangle & |S,1\rangle \\ \end{array}$$

$$U_\Phi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$



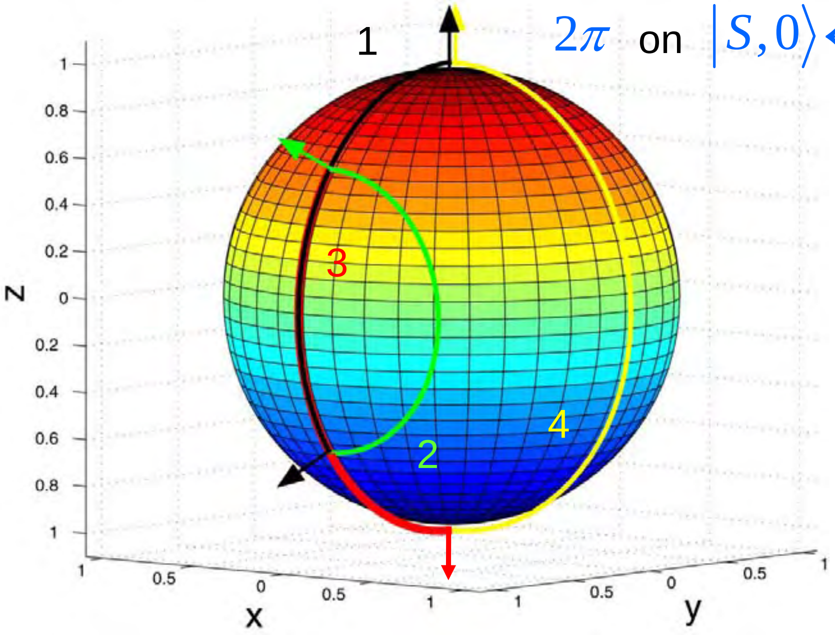
Composite 2π-rotation:



A phase gate with 4 pulses (2π rotation)

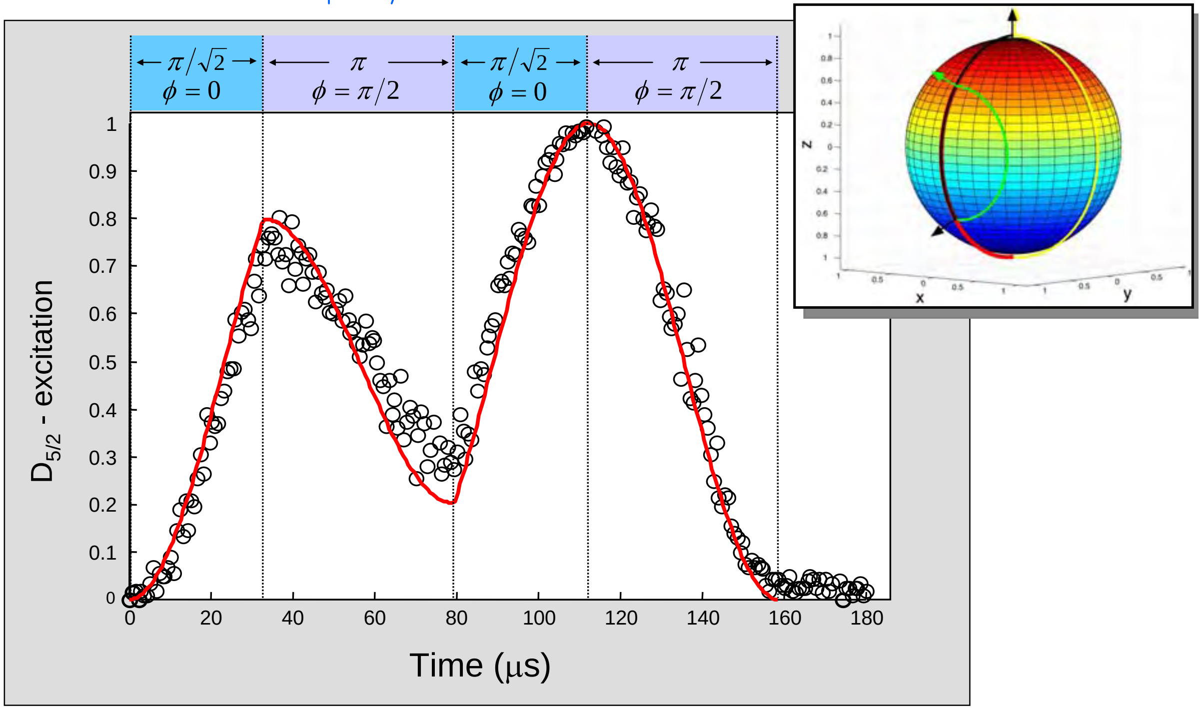
$$R(\theta,\phi) = R_1^+(\pi,\pi/2)\,R_1^+\left(\pi/\sqrt{2},0\right)R_1^+(\pi,\pi/2)\,R_1^+\left(\pi/\sqrt{2},0\right)$$

2π on $|S,0\rangle \leftrightarrow |D,1\rangle$

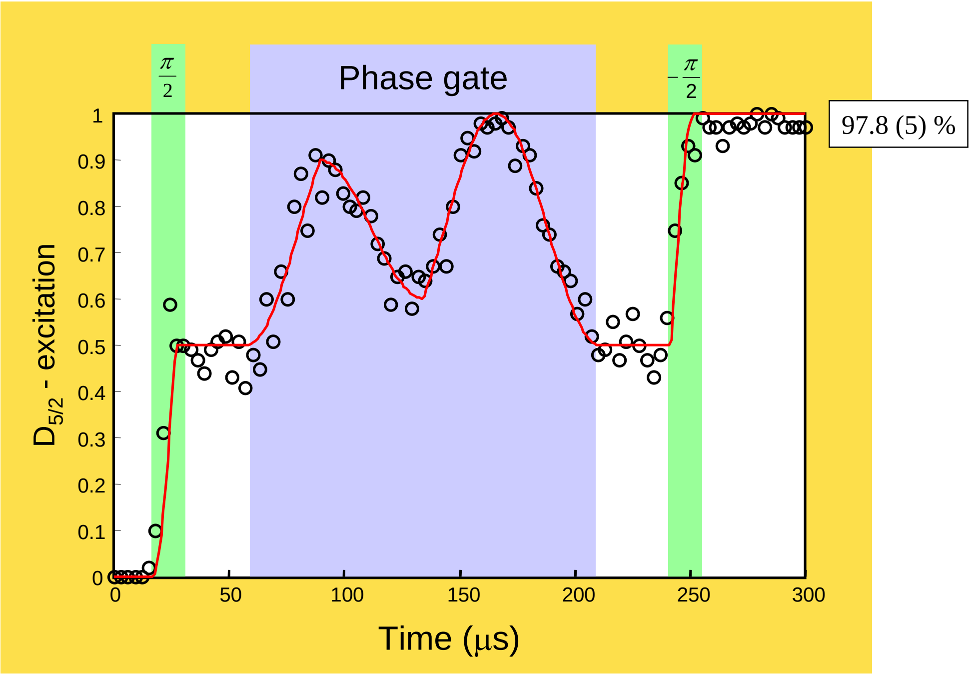


A single ion composite phase gate: Experiment

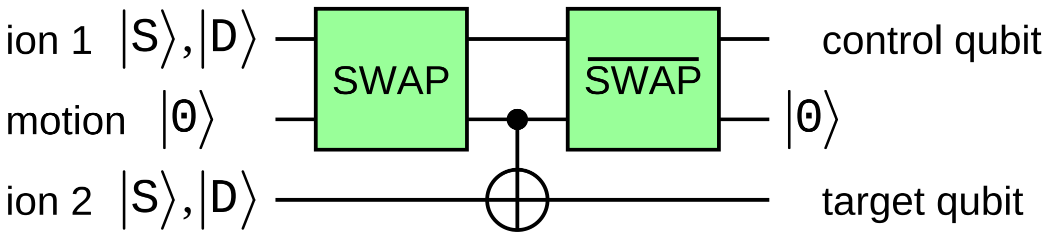
state preparation $\left|S,0\right\rangle$, then application of phase gate pulse sequence



Testing the phase of the phase gate |0,S>



Cirac - Zoller two-ion controlled-NOT operation



ion 1 $|S\rangle,|D\rangle$

motion $|0\rangle$

ion 2 $|S\rangle,|D\rangle$

SWAP

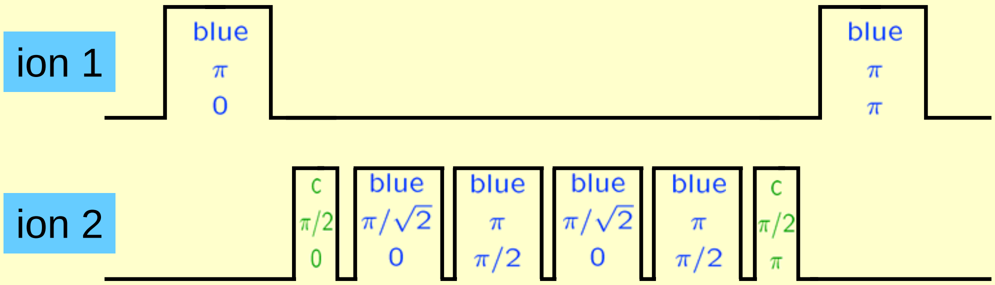
$\overline{\text{SWAP}}$

control qubit

$|0\rangle$

target qubit

pulse sequence:



ion 1

blue π 0

blue π π

ion 2

c $\pi/2$ 0

blue $\pi/\sqrt{2}$ 0

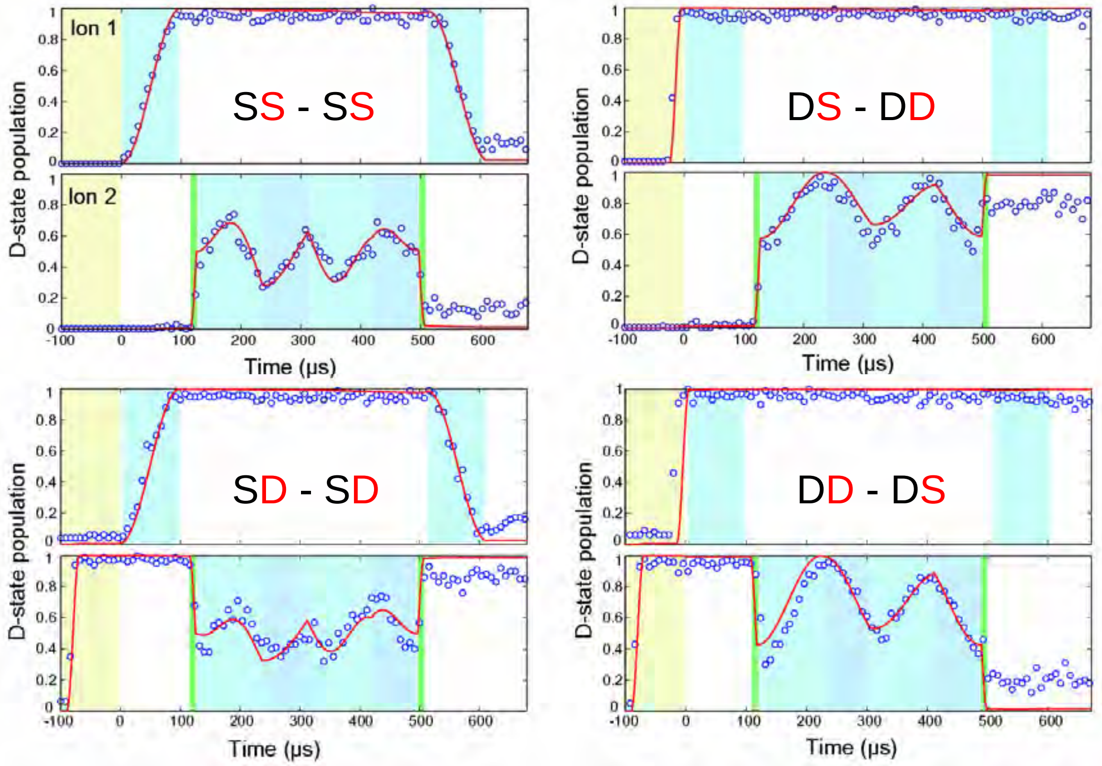
blue π $\pi/2$

blue $\pi/\sqrt{2}$ 0

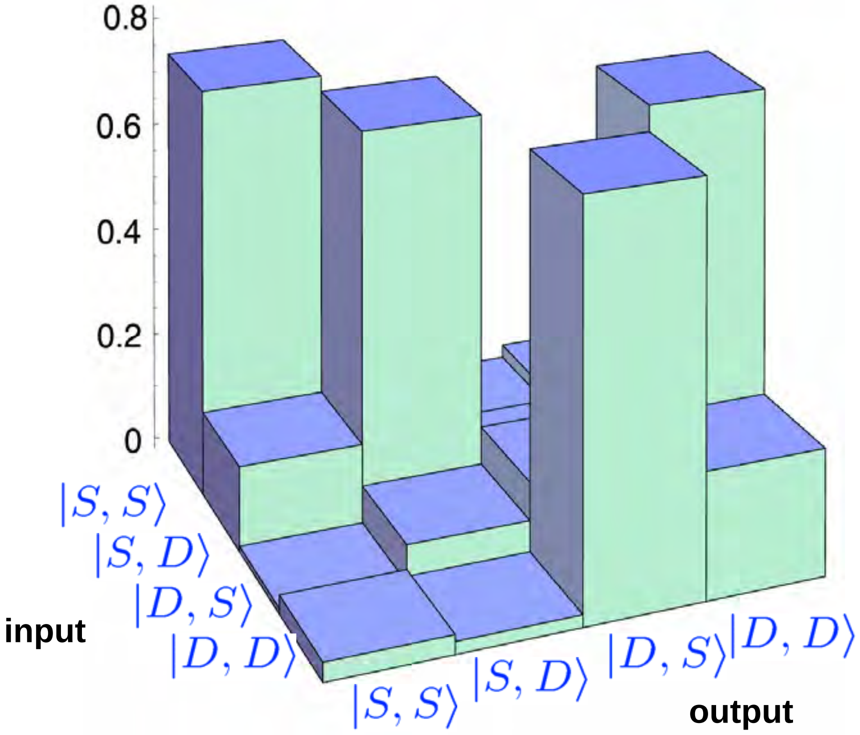
blue π $\pi/2$

c $\pi/2$ π

Cirac – Zoller CNOT gate operation

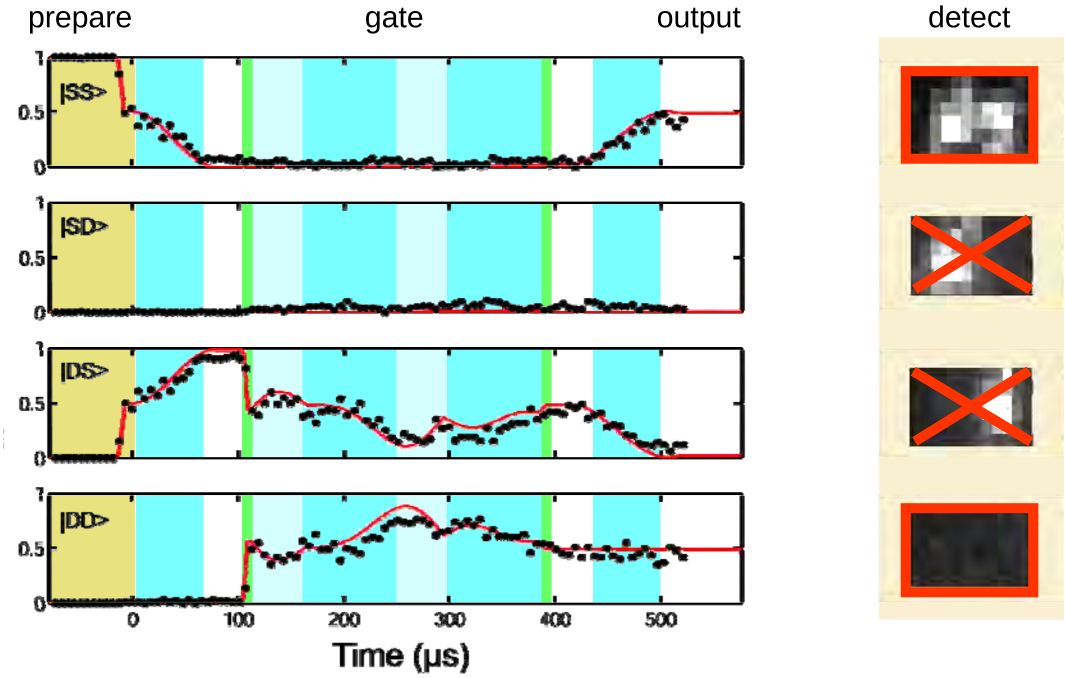


Measured truth table of Cirac-Zoller CNOT operation



Superposition as input to CNOT gate

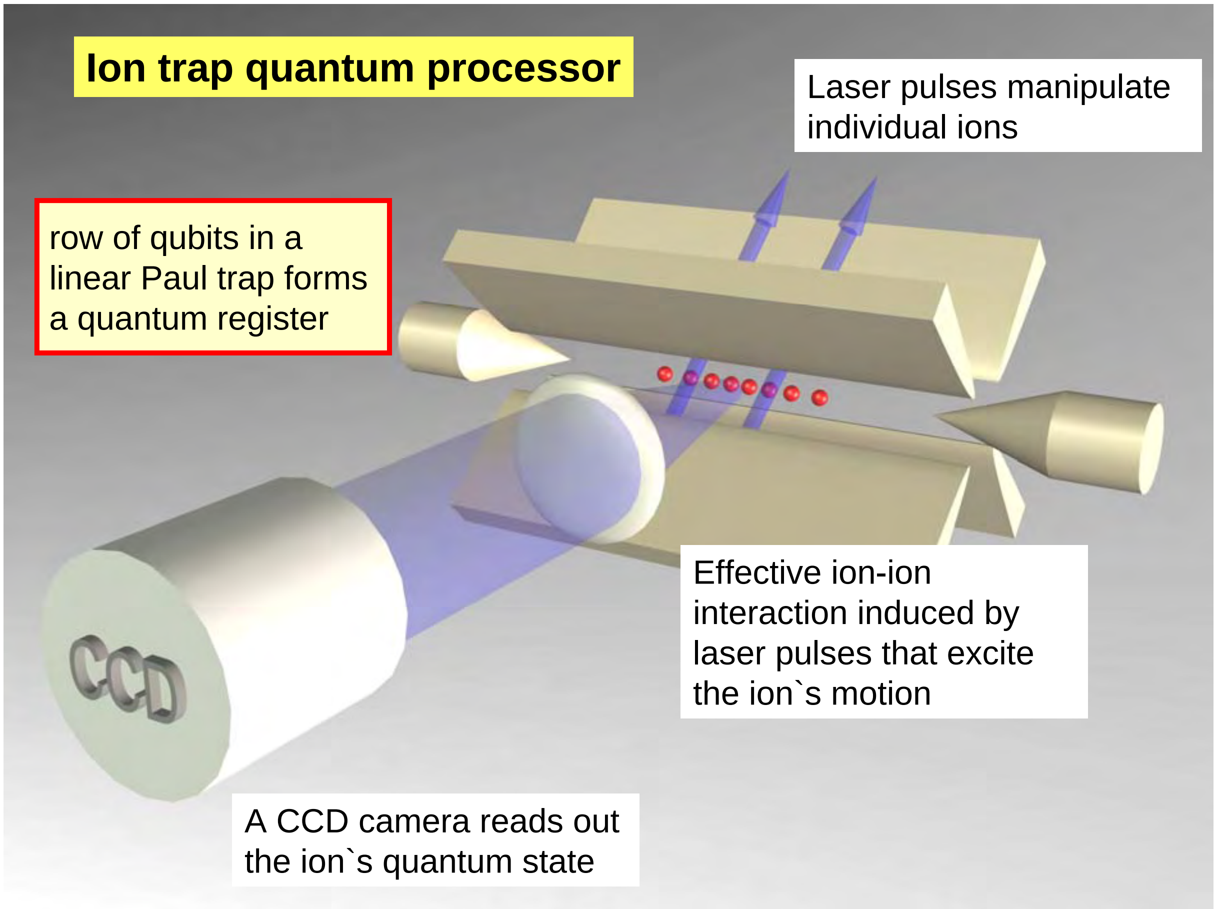
$$|D+S\rangle|S\rangle \longrightarrow |DD\rangle + |SS\rangle$$



Quantum computing with trapped ions (part 2)

Contents

- Two qubit gate
- Decoherence issues
- Implementation of an algorithm
- Scaling of ion trap quantum computers



Ion trap quantum processor

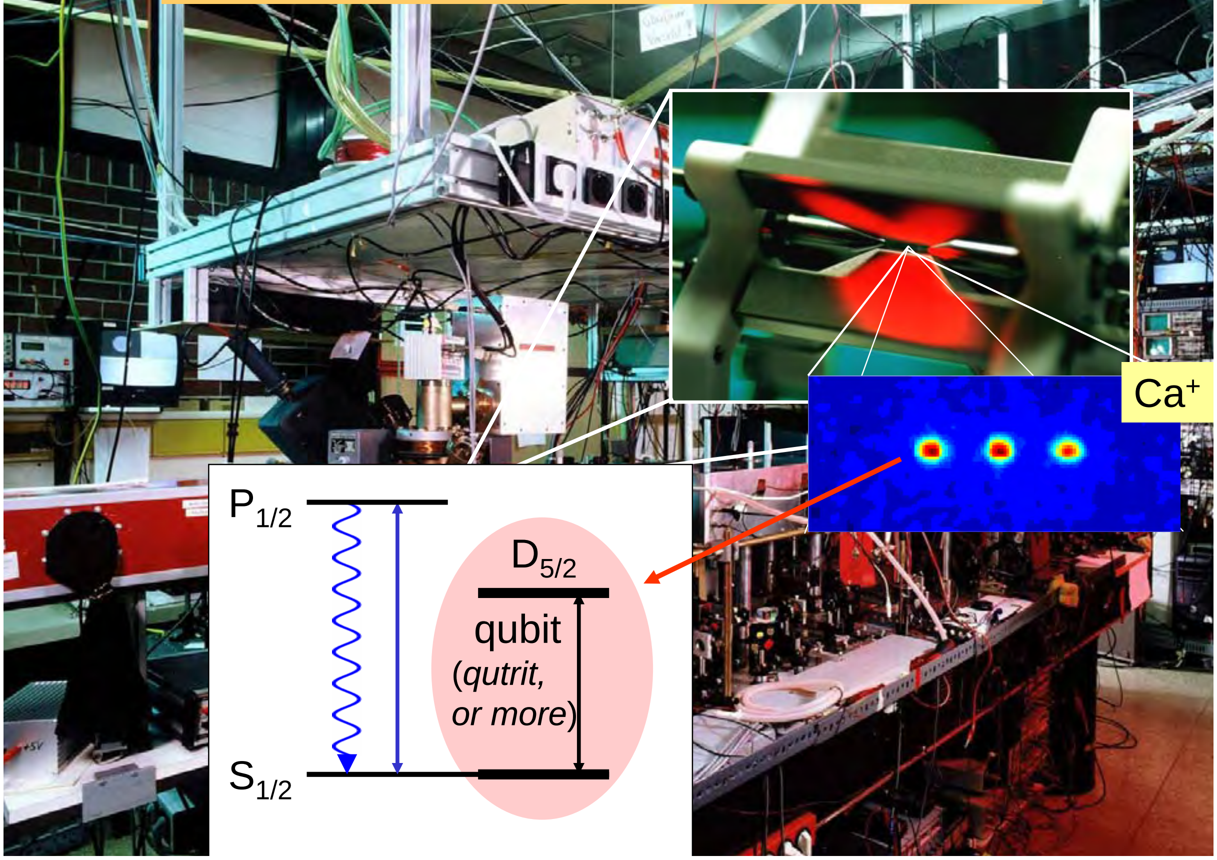
Laser pulses manipulate individual ions

row of qubits in a linear Paul trap forms a quantum register

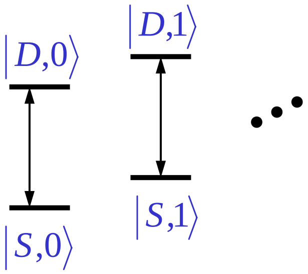
Effective ion-ion interaction induced by laser pulses that excite the ion`s motion

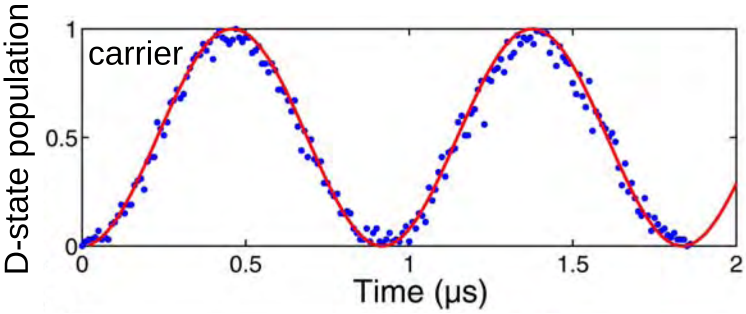
A CCD camera reads out the ion`s quantum state

Experimental setup



Coherent manipulation





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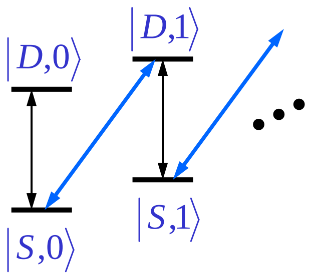
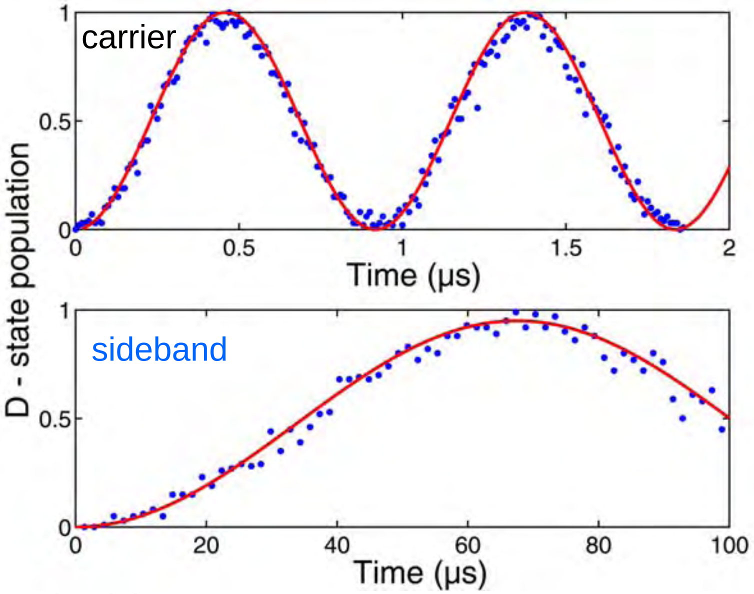


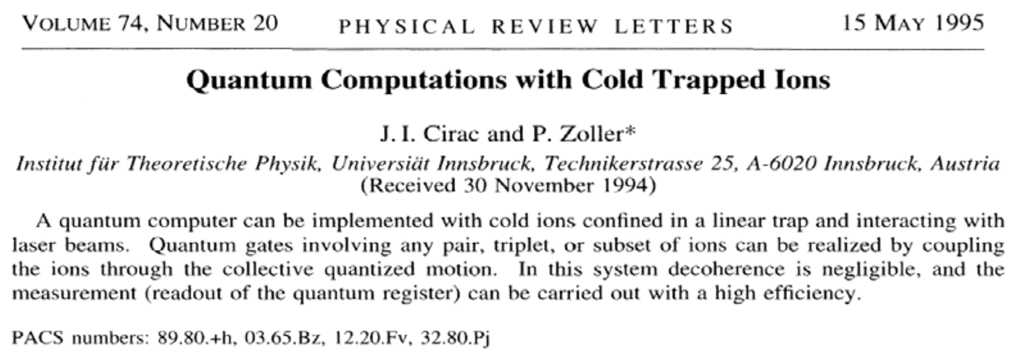


carrier and blue sideband
Rabi oscillations
with Rabi frequencies

Ω and $\eta\Omega$

$\eta = kx_0 \approx 0.02$ is the Lamb-Dicke parameter

Quantum gate proposals with trapped ions



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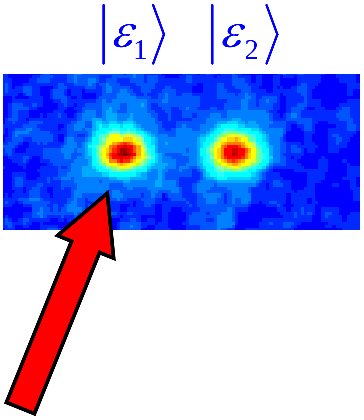

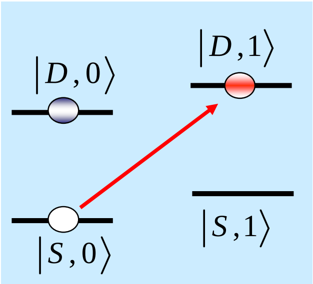

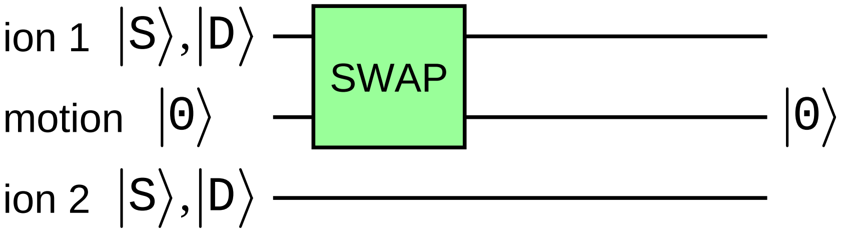

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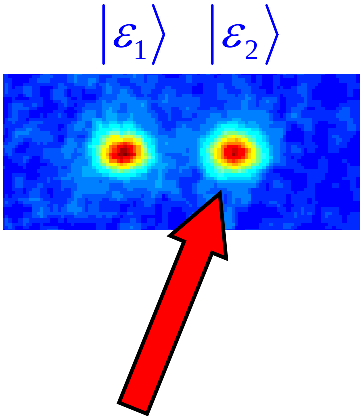

Phase gate using
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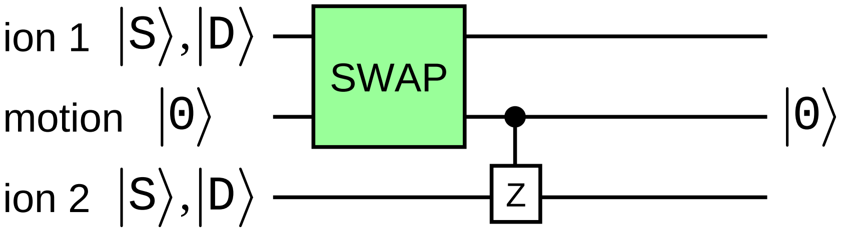

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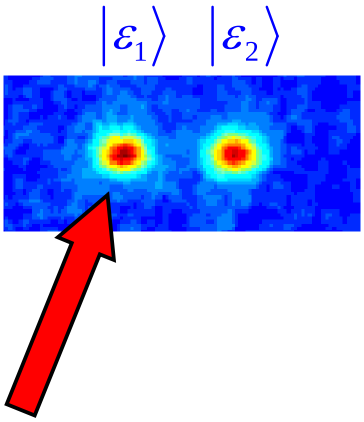

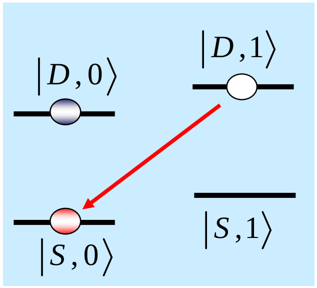

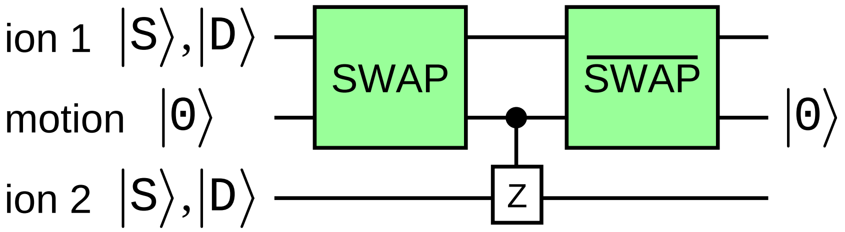

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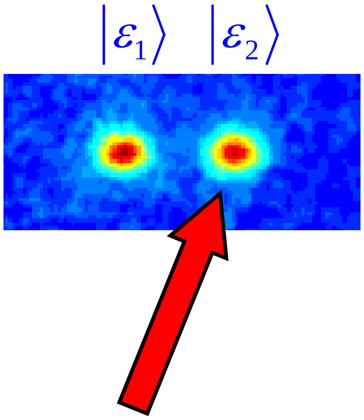

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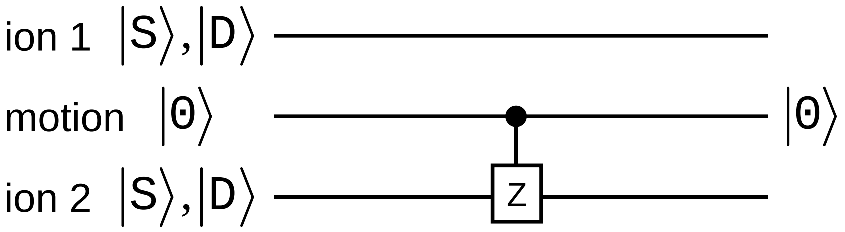
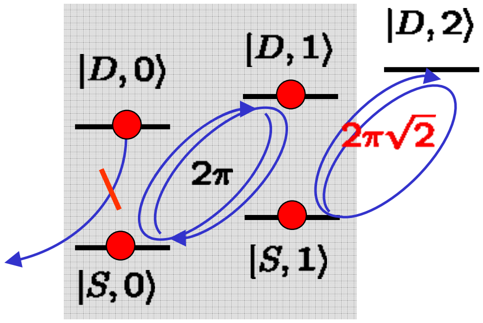


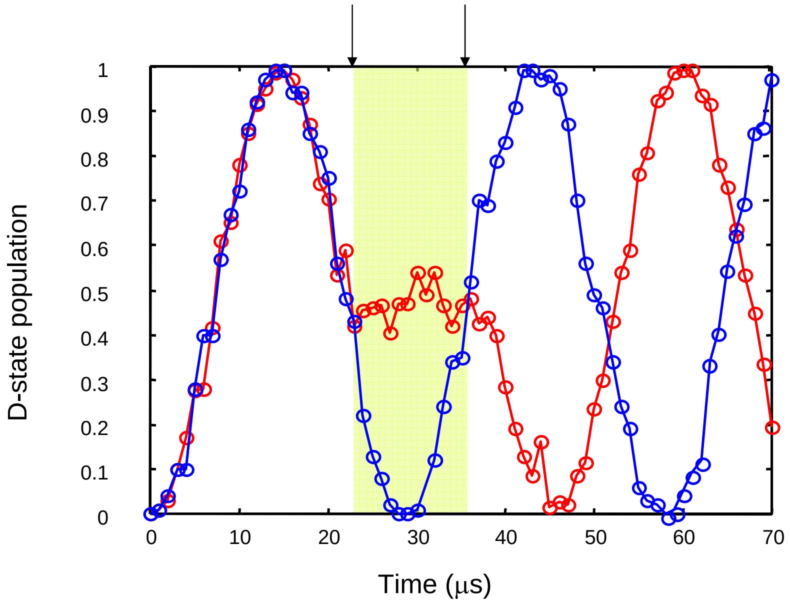
The key step: phase gate

$$\begin{array}{cccccc} & & |D,0\rangle & |S,0\rangle & |D,1\rangle & |S,1\rangle \\ U_\Phi = & \Bigg(& 1 & 0 & 0 & 0 \\ & & 0 & -1 & 0 & 0 \\ & & 0 & 0 & -1 & 0 \\ & & 0 & 0 & 0 & \boxed{?} \Bigg) \end{array}$$



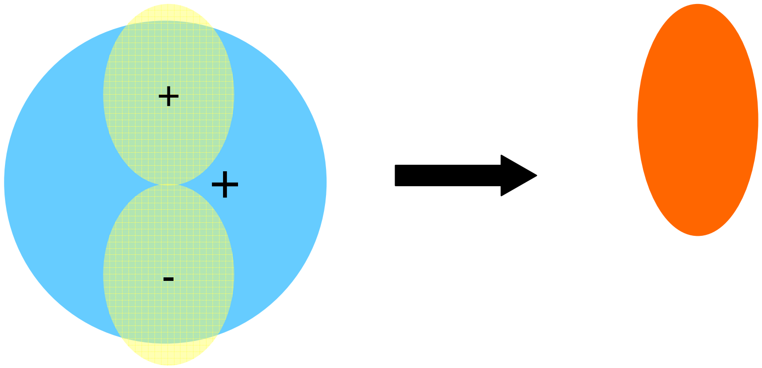
The other type of Rabi oscillations

Phase switched by π/2



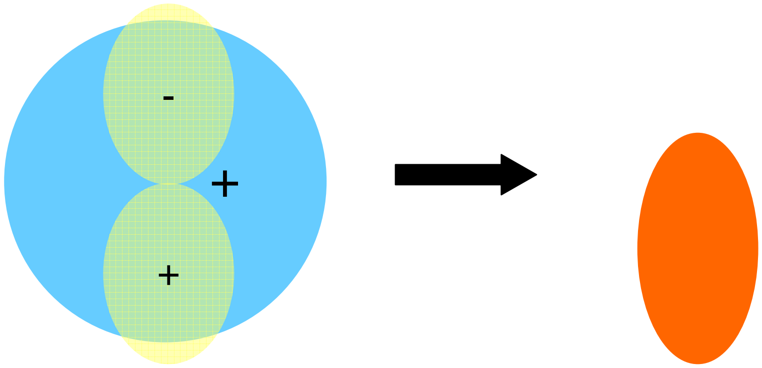
What is this phase ?

The electric field of the laser drives a dipole transition …



What is this phase ?

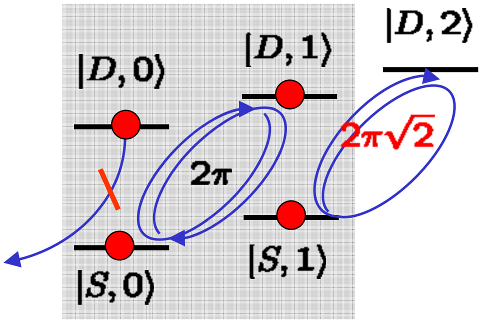
The electric field of the laser drives a dipole transition …



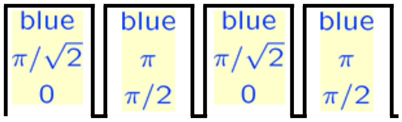
Phase gate

$$U_\Phi = \begin{array}{c} \\ \\ \end{array}\begin{array}{cccc} |D,0\rangle & |S,0\rangle & |D,1\rangle & |S,1\rangle \end{array}$$

$$U_\Phi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$



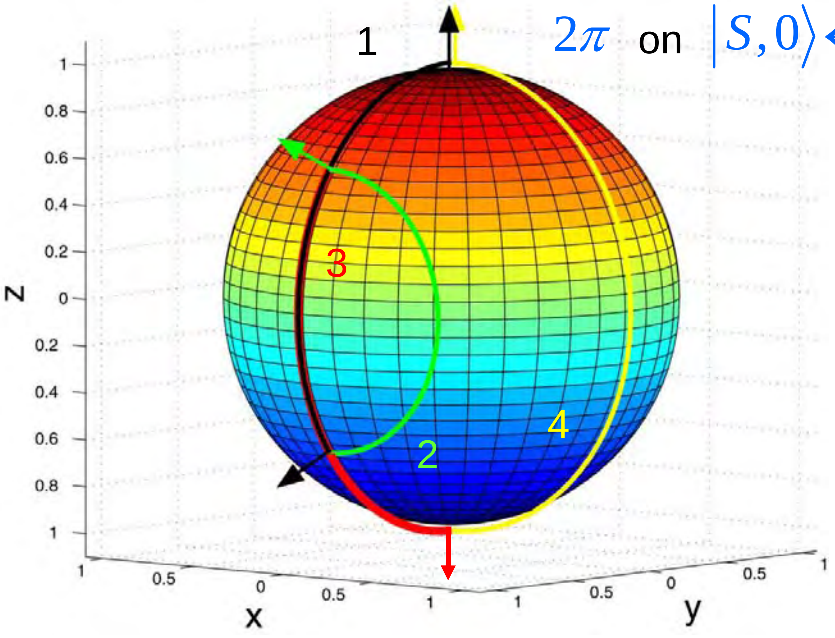
Composite 2π-rotation:



A phase gate with 4 pulses

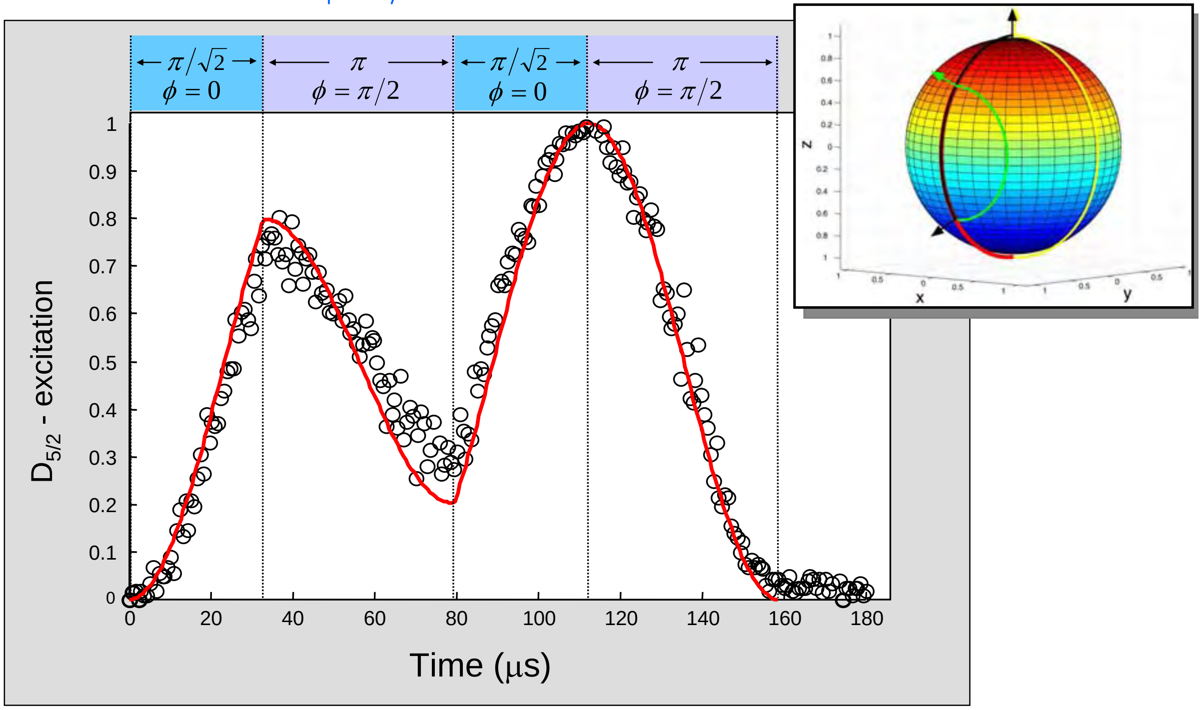
$$R(\theta,\phi) = R_1^+(\pi,\pi/2)\,R_1^+\left(\pi/\sqrt{2},0\right)R_1^+(\pi,\pi/2)\,R_1^+\left(\pi/\sqrt{2},0\right)$$

2π on $|S,0\rangle \leftrightarrow |D,1\rangle$



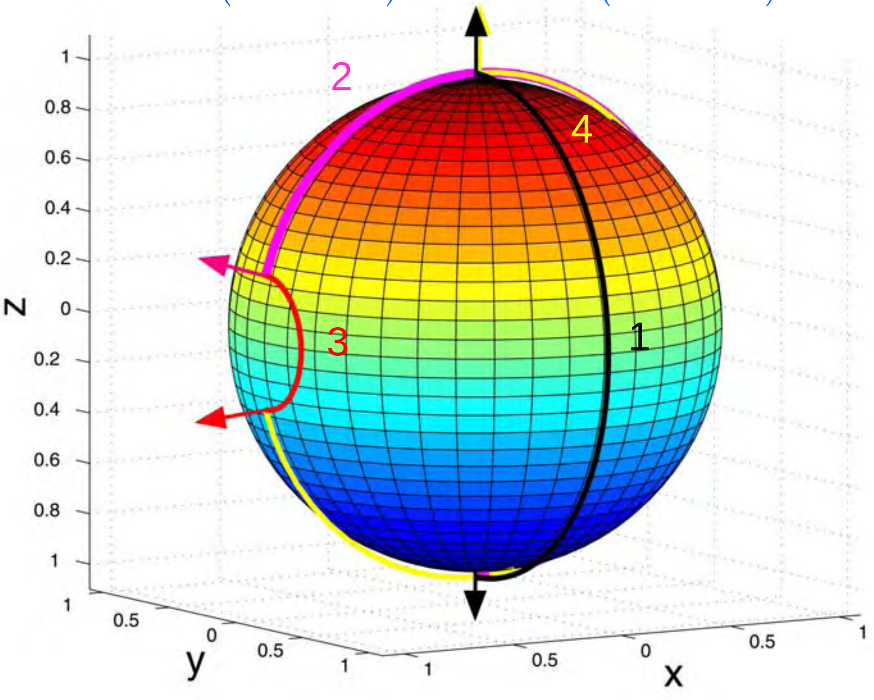
A single ion composite phase gate: Experiment

state preparation $\left|S,0\right\rangle$, then application of phase gate pulse sequence

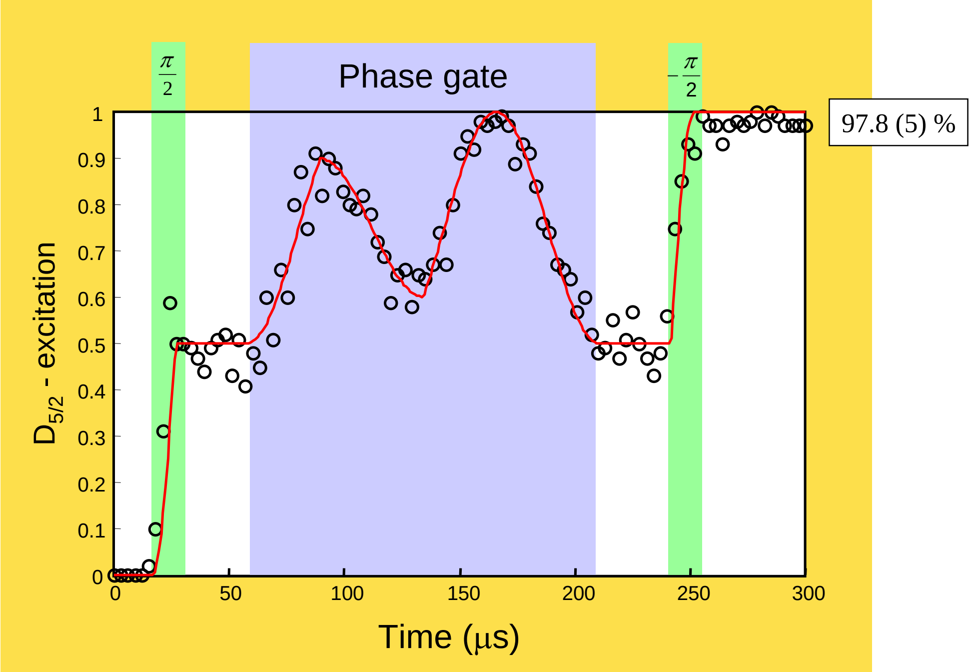


Population of |S,1> - |D,2> remains unaffected

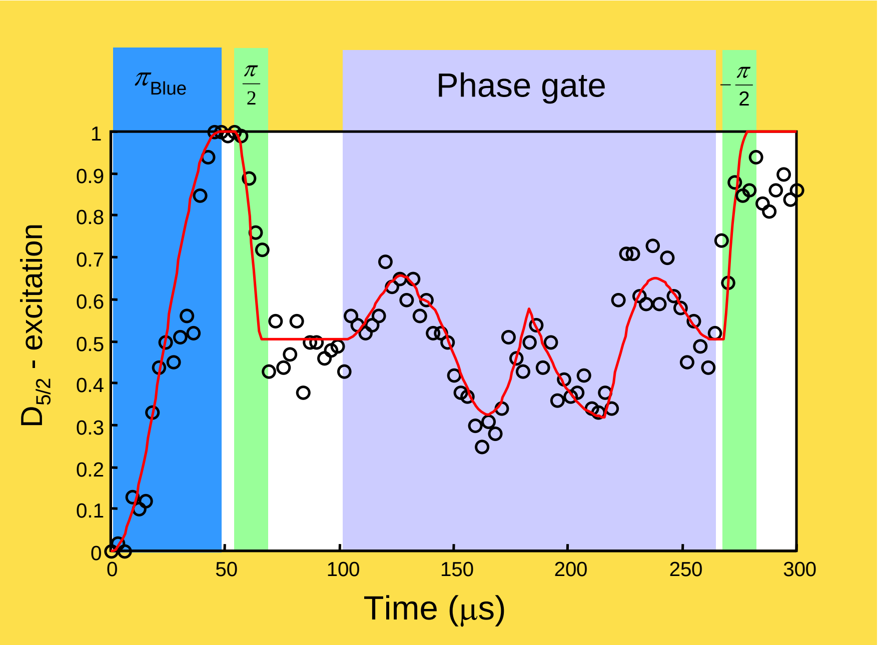
$$R(\theta,\phi) = R_1^+\left(\pi\sqrt{2},\pi/2\right)R_1^+\left(\pi,0\right)R_1^+\left(\pi\sqrt{2},\pi/2\right)R_1^+\left(\pi,0\right)$$



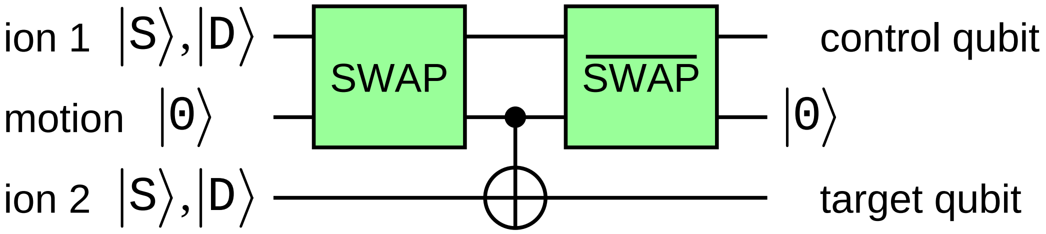
Testing the phase of the phase gate |0,S>



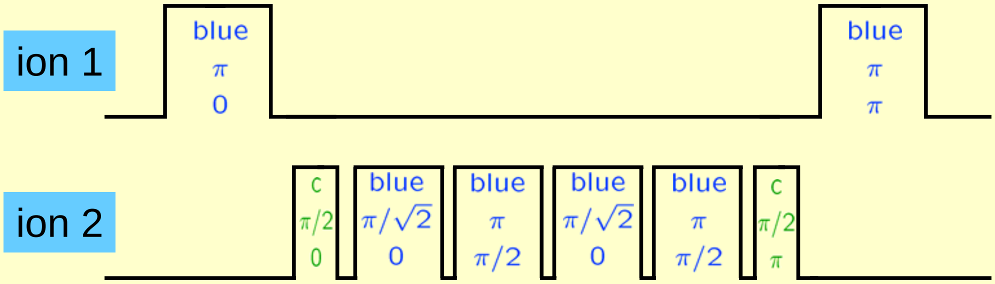
Phase gate with starting in |D,1>



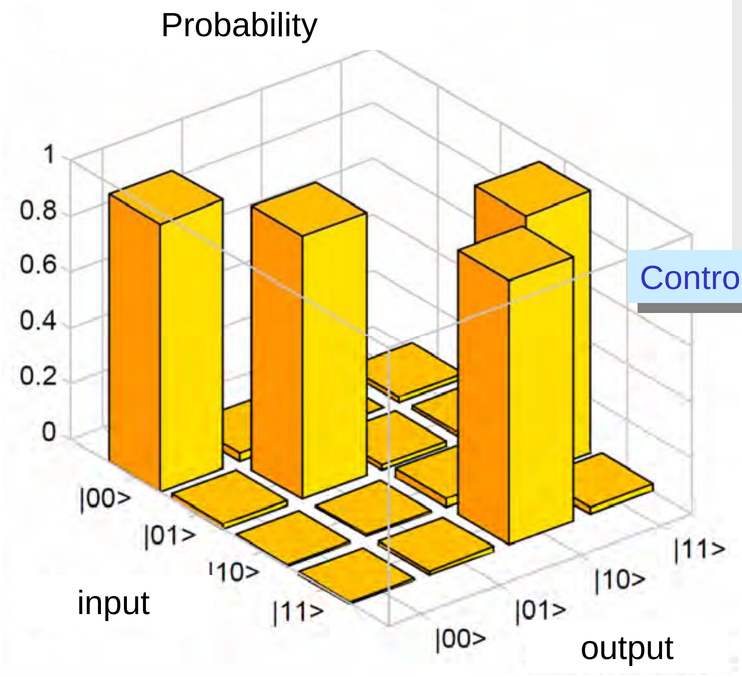
Cirac - Zoller two-ion controlled-NOT operation



pulse sequence:



Truth table of the CNOT



$|0\rangle|0\rangle \rightarrow |0\rangle|0\rangle$

$|0\rangle|1\rangle \rightarrow |0\rangle|1\rangle$

$|1\rangle|0\rangle \rightarrow |1\rangle|1\rangle$

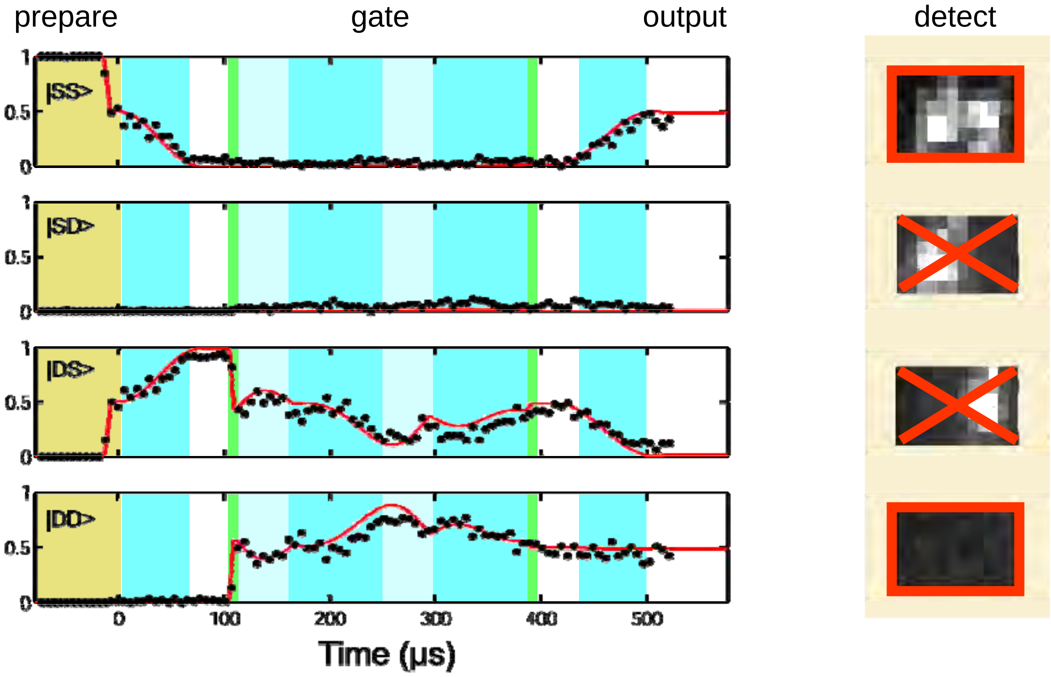
$|1\rangle|1\rangle \rightarrow |1\rangle|0\rangle$

Control bit

Target bit

Superposition as input to CNOT gate

$$|D+S\rangle|S\rangle \longrightarrow |DD\rangle + |SS\rangle$$

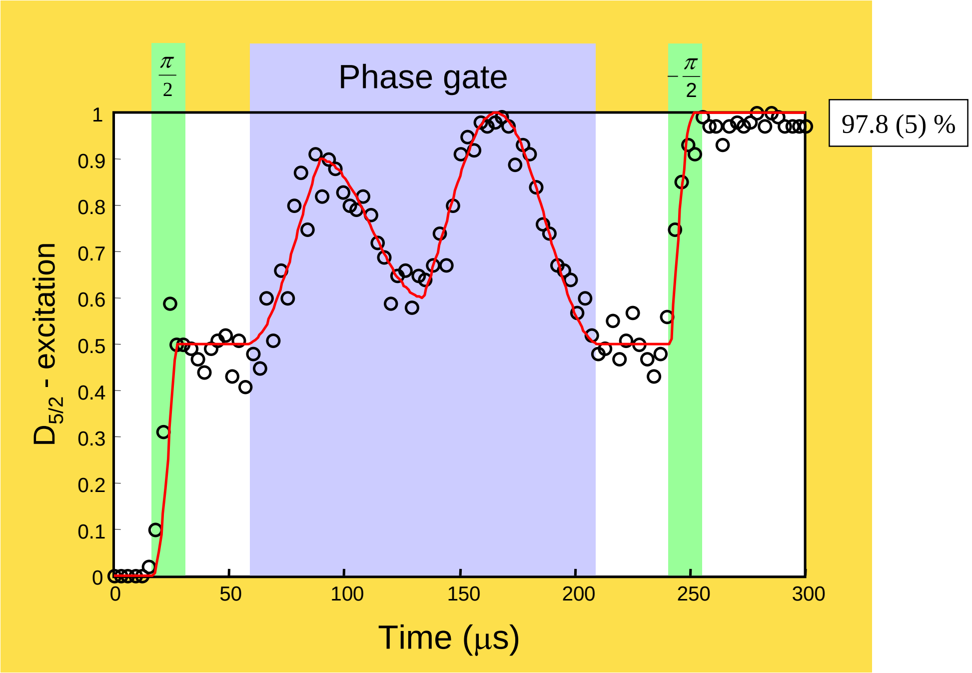


- Two qubit gate
- Decoherence issues
- Implementation of an algorithm
- Scaling of ion trap quantum computers

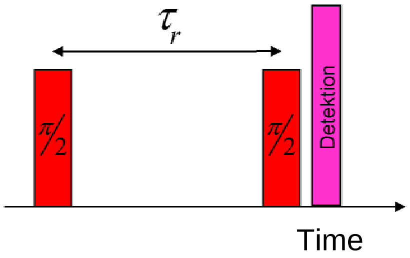
Error budget for Cirac-Zoller CNOT

Error source	Magnitude	Fidelity loss
Laser frequency noise (Phase coherence)	} ~ 3 Hz (FWHM)	~ 1 %
Residual thermal excitation	$<n>_{bus}$ < 0.01 $<n>_{spec}$ = 6	< 1 % 0.4 %
Laser intensity noise	5 % peak to peak	0.3 %
Addressing error (can be corrected for partially)	5 % in Rabi frequency (at neighbouring ion)	3 %
Laser detuning error	~ 500 Hz (FWHM)	~ 3 %
Total	November 2006	9 %

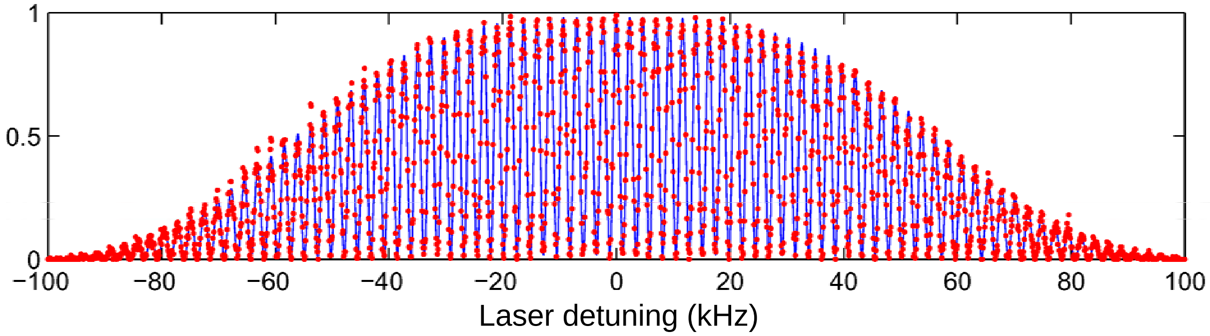
Testing the phase of the phase gate |0,S>



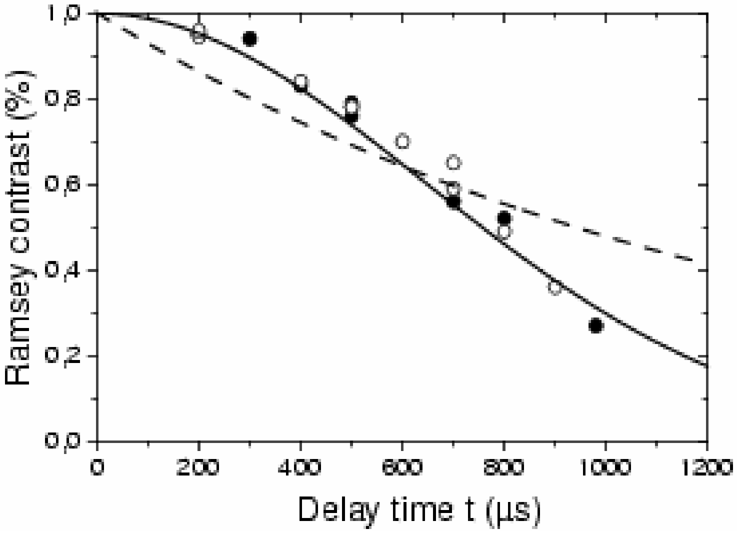
Ramsey experiment



Quantum algorithms can be viewed as generalized Ramsey experiments!



Phase coherence



⇒ Gaussian modell yields a coherence time of 0.9 ms
2005: 2 ms are more typical.
2007: 8 ms are more typical.

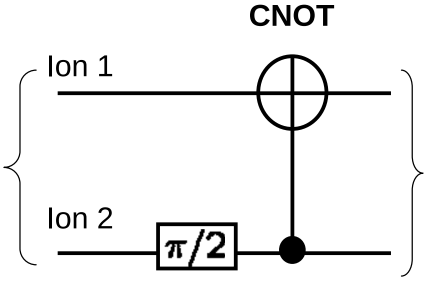
Creating entangled states with a CNOT

$|SS\rangle$

$|DS\rangle$

$|SD\rangle$

$|DD\rangle$



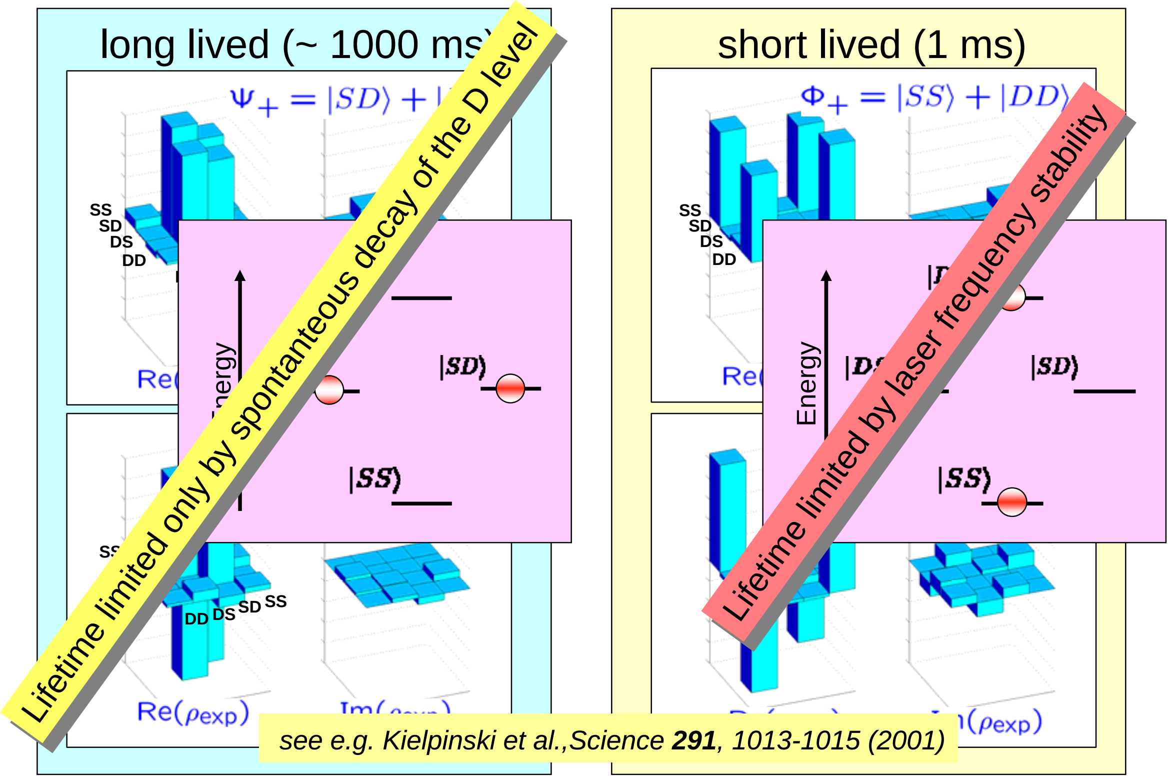
$$\Phi_{+} = \frac{1}{\sqrt{2}}\left(|SS\rangle + |DD\rangle\right)$$

$$\Psi_{+} = \frac{1}{\sqrt{2}}\left(|SD\rangle + |DS\rangle\right)$$

$$\Phi_{-} = \frac{1}{\sqrt{2}}\left(|SS\rangle - |DD\rangle\right)$$

$$\Psi_{-} = \frac{1}{\sqrt{2}}\left(|SD\rangle - |DS\rangle\right)$$

Decoherence of entangled states



long lived (~ 1000 ms)

short lived (1 ms)

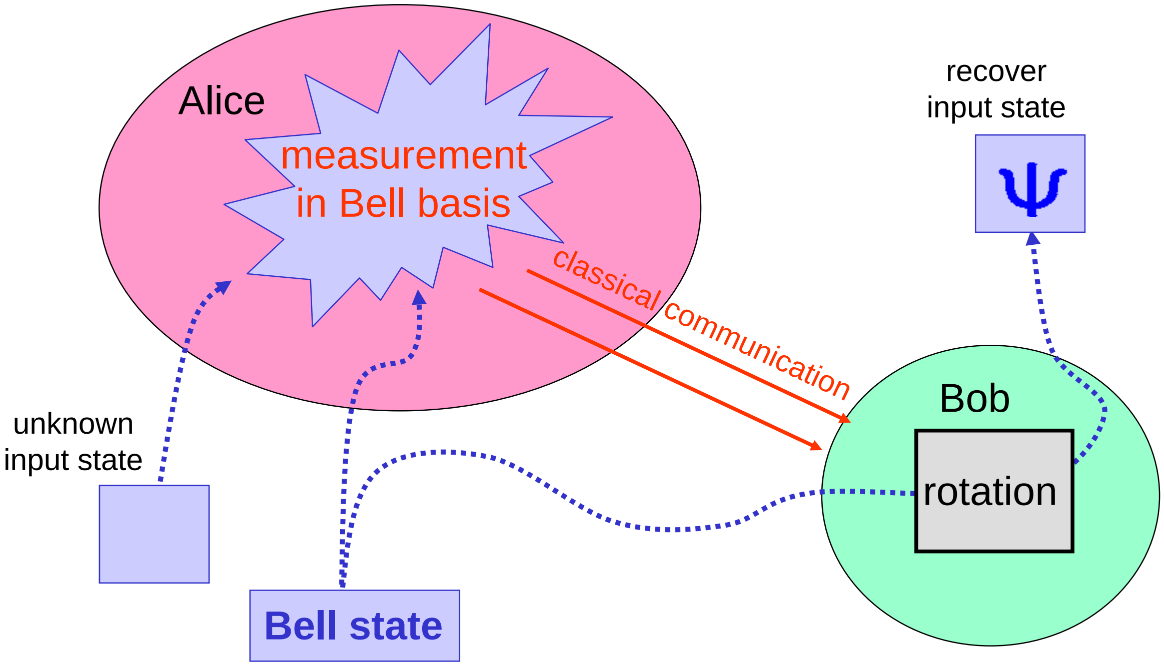
Lifetime limited only by spontanteous decay of the D level

Lifetime limited by laser frequency stability

see e.g. Kielpinski et al.,Science ***291****, 1013-1015 (2001)*

- Two qubit gate
- Decoherence issues
- Implementation of an algorithm
- Scaling of ion trap quantum computers

Teleportation



Alice

measurement in Bell basis

classical communication

recover input state

Ψ

Bob

rotation

unknown input state

Bell state

Bennett et *al.*, *Phys. Rev. Lett. 70, 1895 (1993)*

Principle of Teleportation

Blue=Alice
Red=Bob

unknown state $|\phi\rangle$

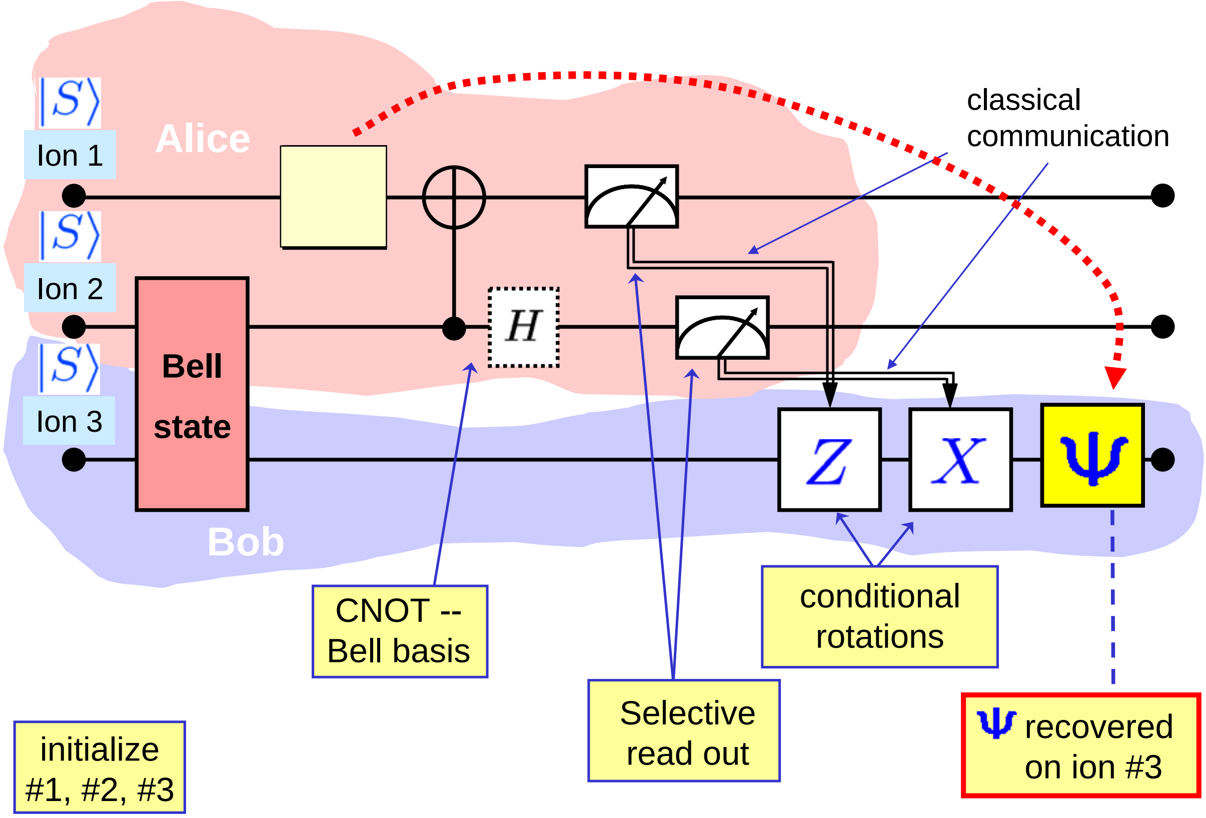
EPR pair $|\Psi_-\rangle$

$$(\alpha|D\rangle_1 + \beta|S\rangle_1)(|SD\rangle_{23} - |DS\rangle_{23}) = \frac{1}{2}[\,|\Psi_-\rangle_{12}(\quad -I|\phi\rangle_3)$$
$$+|\Psi_+\rangle_{12}(\ i\exp(i\frac{\pi}{2}\sigma_z)|\phi\rangle_3)$$
$$+|\Phi_-\rangle_{12}(-i\exp(i\frac{\pi}{2}\sigma_x)|\phi\rangle_3)$$
$$+|\Phi_+\rangle_{12}(-\exp(i\frac{\pi}{2}\sigma_y)|\phi\rangle_3)]$$

Alice: Measurement in Bell state basis

$\{\Psi_-, \Psi_+, \Phi_+, \Phi_-\}$

Implementation of the teleportation protocol



classical communication

CNOT -- Bell basis

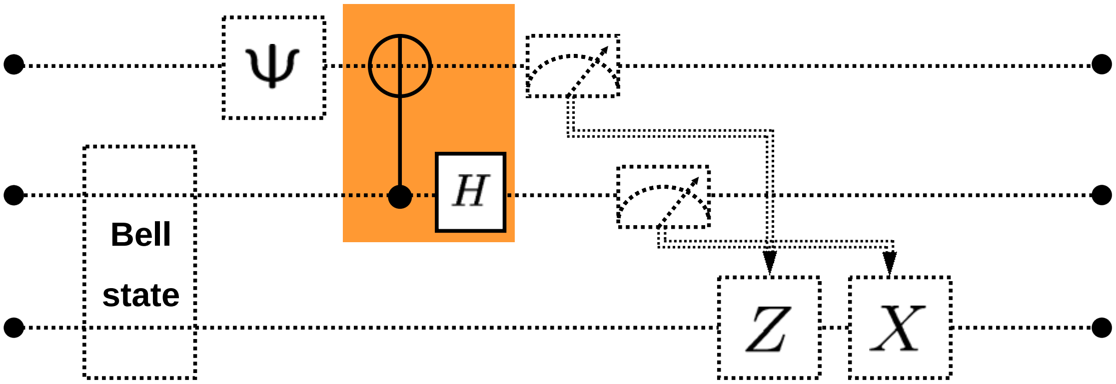
Selective read out

conditional rotations

Ψ recovered on ion #3

initialize #1, #2, #3

Teleportation: Rotate into the Bell basis



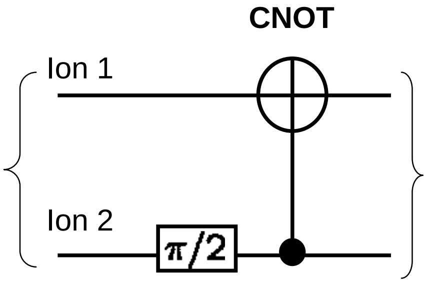
Creating entangled states with a CNOT

$|SS\rangle$

$|DS\rangle$

$|SD\rangle$

$|DD\rangle$



$$\Phi_+ = \frac{1}{\sqrt{2}}\left(|SS\rangle + |DD\rangle\right)$$

$$\Psi_+ = \frac{1}{\sqrt{2}}\left(|SD\rangle + |DS\rangle\right)$$

$$\Phi_- = \frac{1}{\sqrt{2}}\left(|SS\rangle - |DD\rangle\right)$$

$$\Psi_- = \frac{1}{\sqrt{2}}\left(|SD\rangle - |DS\rangle\right)$$

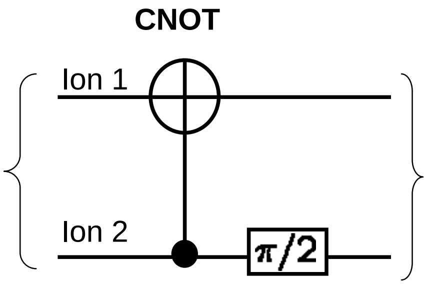
And now backwards ...

$$\Phi_+ = \frac{1}{\sqrt{2}}\left(|SS\rangle + |DD\rangle\right)$$

$$\Psi_+ = \frac{1}{\sqrt{2}}\left(|SD\rangle + |DS\rangle\right)$$

$$\Phi_- = \frac{1}{\sqrt{2}}\left(|SS\rangle - |DD\rangle\right)$$

$$\Psi_- = \frac{1}{\sqrt{2}}\left(|SD\rangle - |DS\rangle\right)$$



$|SS\rangle$

$|DS\rangle$

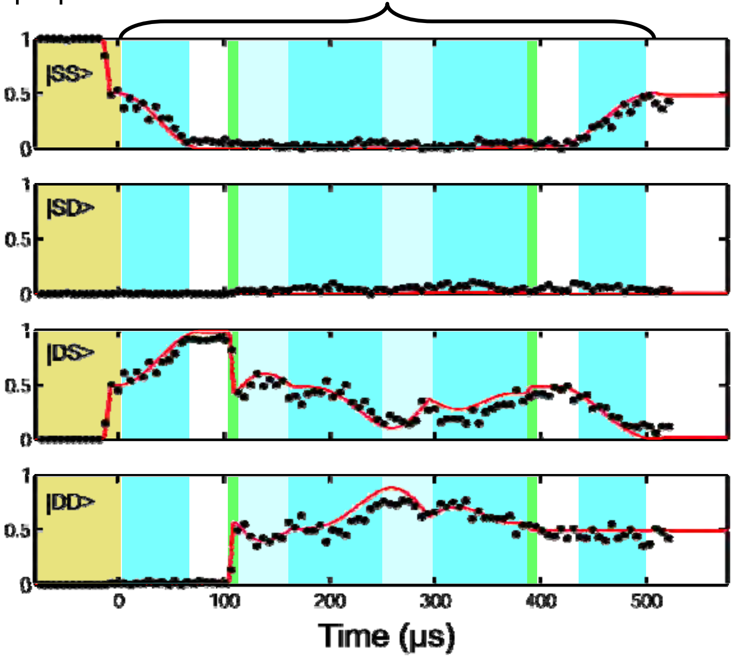
$|SD\rangle$

$|DD\rangle$

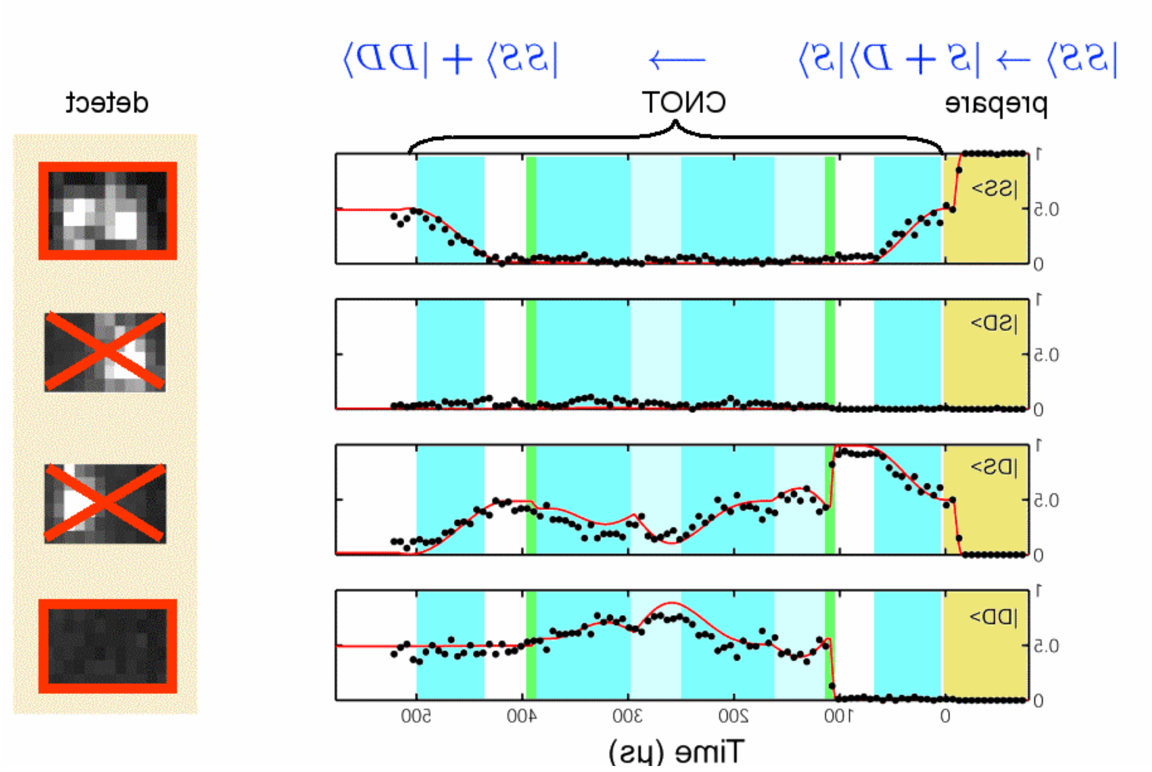
Superposition as input to a CNOT gate

$$|SS\rangle \rightarrow |S+D\rangle|S\rangle \quad \longrightarrow \quad |SS\rangle + |DD\rangle$$

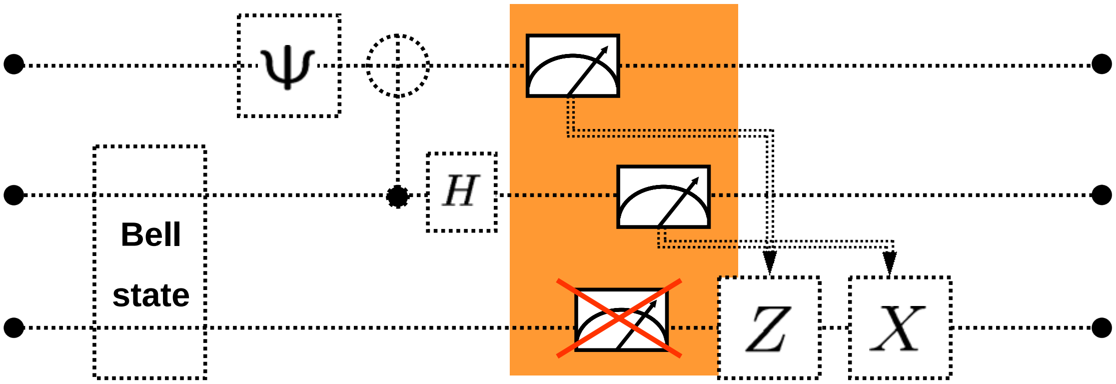
prepare CNOT detect



And now backwards ...



Teleportation: Selective read-out of a quantum register



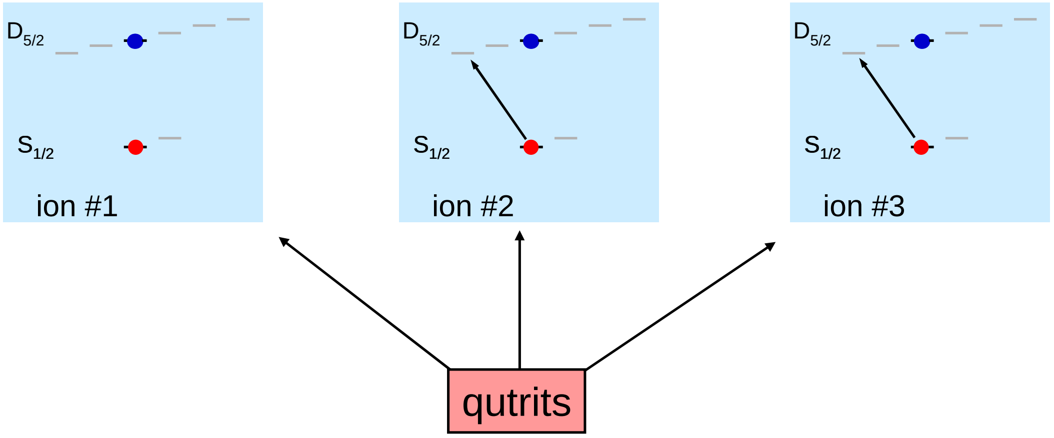
Selective read-out of the ion string

We want to read out some ions without disturbing the coherence of the neighbouring ions.
This is required for
- most error correction protocols.
- teleportation.
- having fun with the ions.

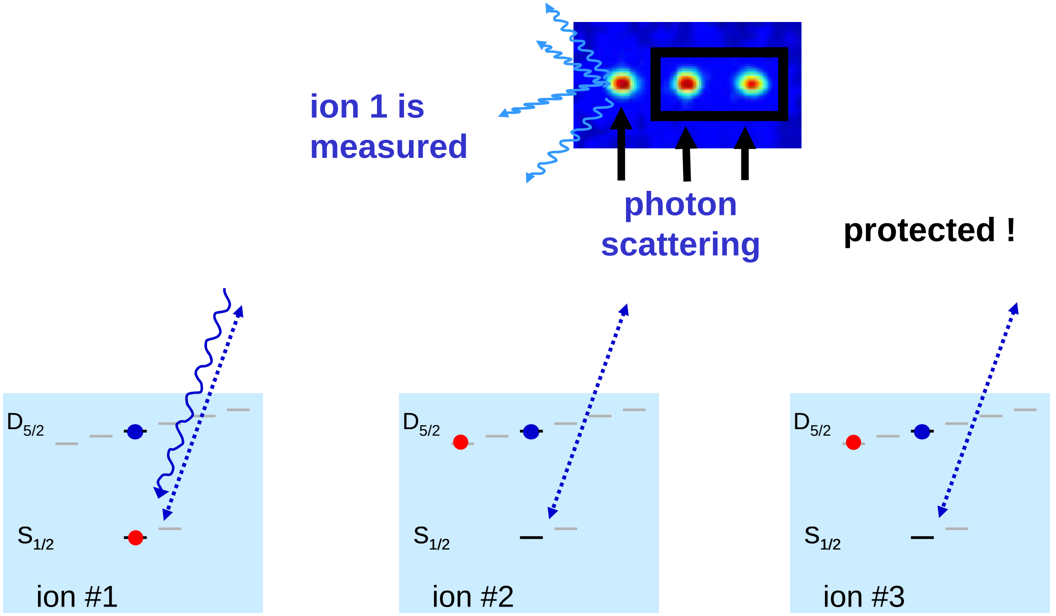
Where is the problem?

1. The ions are spaced by about 5 μm.
2. One photon projects an ion onto the measurement basis.

Selective read out



Selective read out



ion 1 is measured

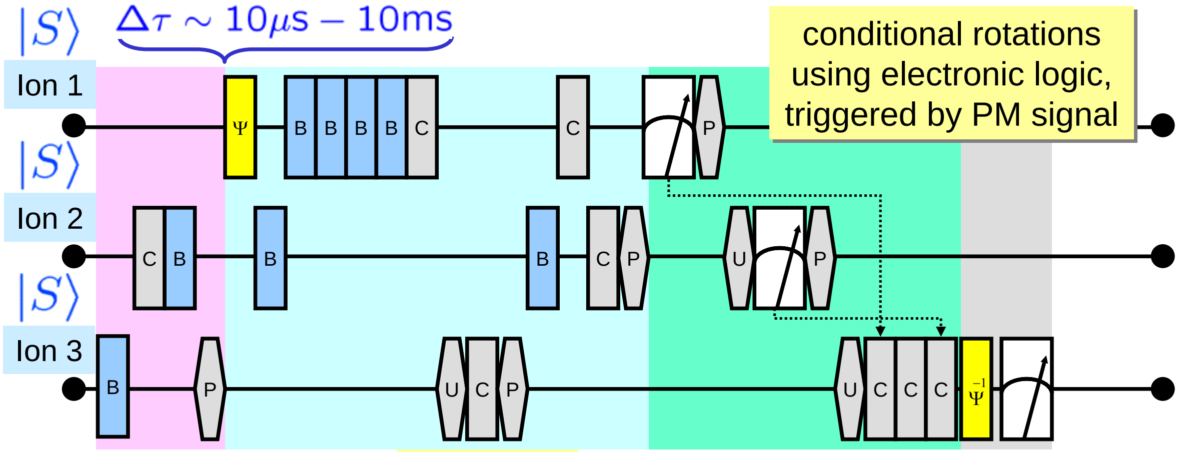
photon scattering

protected !

$$|S\blacksquare\rangle + |D\blacksquare\rangle + |D\blacksquare\rangle$$

detect quantum state of ion #1 only

Teleportation protocol, details



conditional rotations using electronic logic, triggered by PM signal

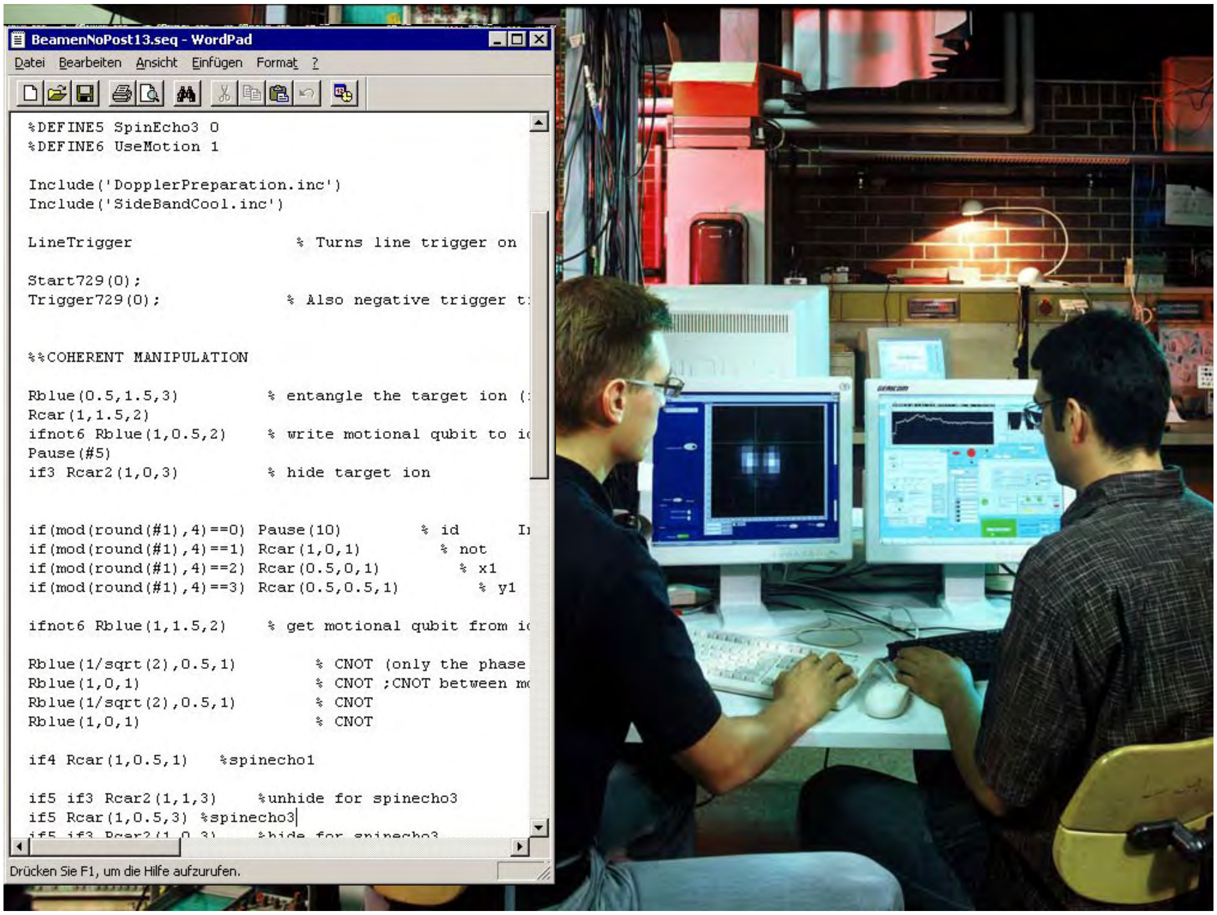
spin echo sequence

B blue sideband pulses $R^{+}(\theta,\varphi)$

C carrier pulses $R(\theta,\varphi)$

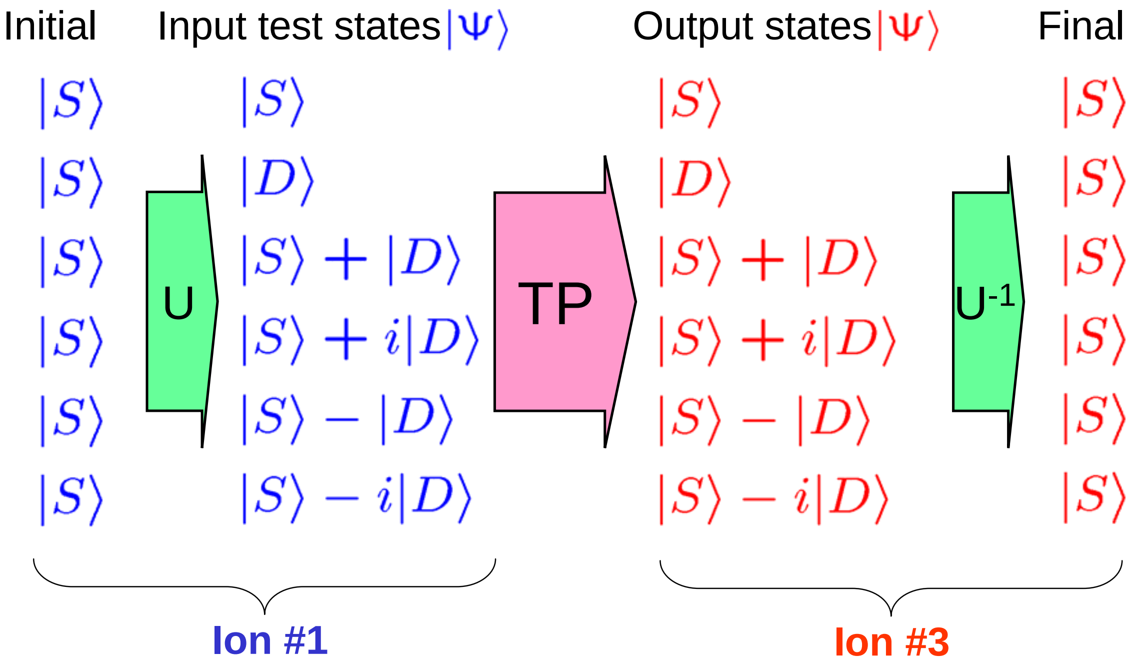
$\tau \approx 2$ ms

full sequence:
26 pulses + 2 measurements



```
%DEFINE5 SpinEcho3 0
%DEFINE6 UseMotion 1

Include('DopplerPreparation.inc')
Include('SideBandCool.inc')

LineTrigger                 % Turns line trigger on

Start729(0);
Trigger729(0);             % Also negative trigger t


%%COHERENT MANIPULATION

Rblue(0.5,1.5,3)          % entangle the target ion (
Rcar(1,1.5,2)
ifnot6 Rblue(1,0.5,2)     % write motional qubit to i
Pause(#5)
if3 Rcar2(1,0,3)          % hide target ion


if(mod(round(#1),4)==0) Pause(10)        % id      I
if(mod(round(#1),4)==1) Rcar(1,0,1)        % not
if(mod(round(#1),4)==2) Rcar(0.5,0,1)        % x1
if(mod(round(#1),4)==3) Rcar(0.5,0.5,1)        % y1

ifnot6 Rblue(1,1.5,2)     % get motional qubit from i

Rblue(1/sqrt(2),0.5,1)         % CNOT (only the phase
Rblue(1,0,1)                   % CNOT ;CNOT between m
Rblue(1/sqrt(2),0.5,1)         % CNOT
Rblue(1,0,1)                   % CNOT

if4 Rcar(1,0.5,1)   %spinecho1

if5 if3 Rcar2(1,1,3)    %unhide for spinecho3
if5 Rcar(1,0.5,3) %spinecho3
if5 if3 Rcar2(1,0,3)    %hide for spinecho3
```

Teleportation procedure, analysis

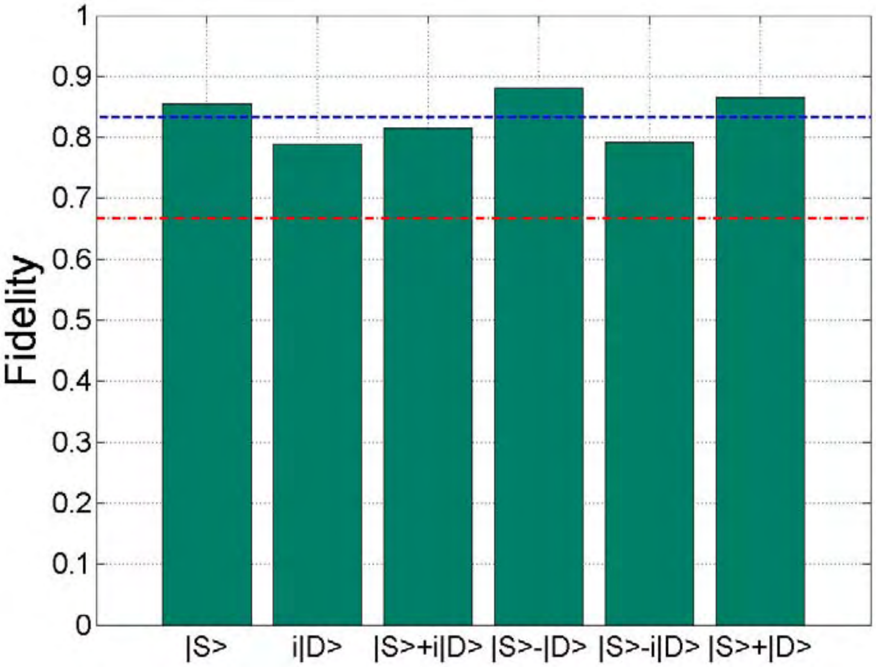


Initial	Input test states $|\Psi\rangle$	Output states $|\Psi\rangle$	Final
$|S\rangle$	$|S\rangle$	$|S\rangle$	$|S\rangle$
$|S\rangle$	$|D\rangle$	$|D\rangle$	$|S\rangle$
$|S\rangle$	$|S\rangle + |D\rangle$	$|S\rangle + |D\rangle$	$|S\rangle$
$|S\rangle$	$|S\rangle + i|D\rangle$	$|S\rangle + i|D\rangle$	$|S\rangle$
$|S\rangle$	$|S\rangle - |D\rangle$	$|S\rangle - |D\rangle$	$|S\rangle$
$|S\rangle$	$|S\rangle - i|D\rangle$	$|S\rangle - i|D\rangle$	$|S\rangle$

Ion #1

Ion #3

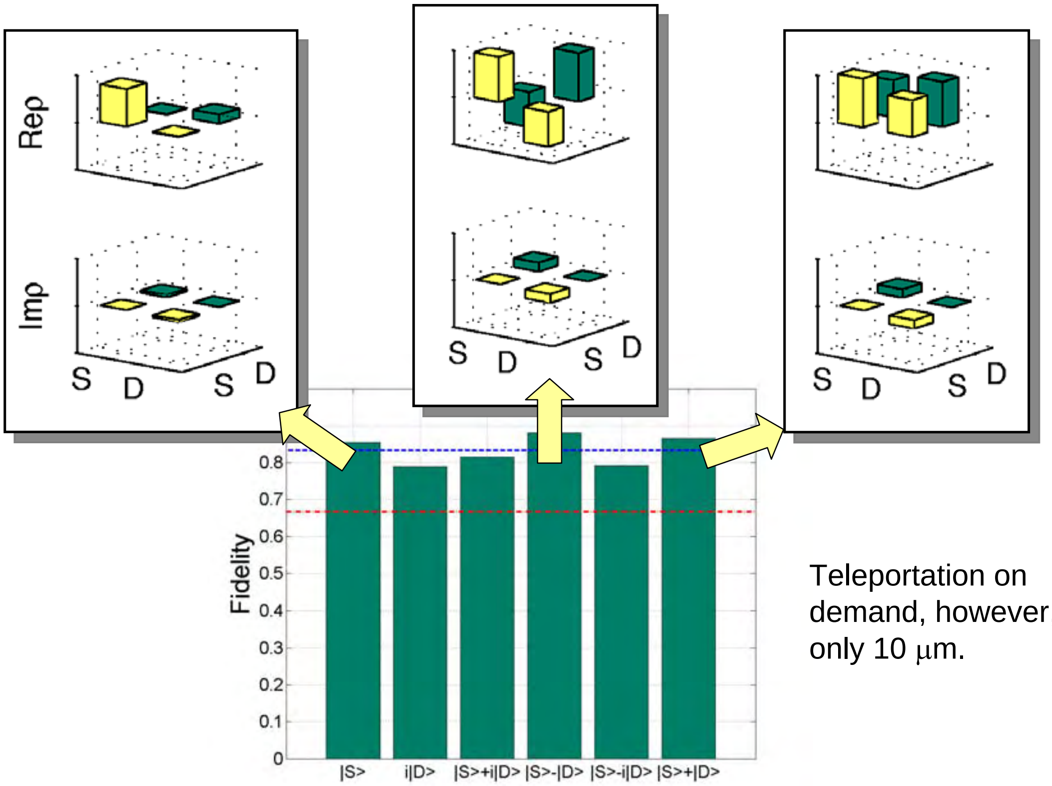
Quantum teleportation on demand



Fidelity: 0.83

Classical
threshold

Process tomography of teleportation



Teleportation on
demand, however,
only 10 μm.

What is teleportation good for?

Supose you can somehow entangle two qubits in different quantum computers with some dirty trick.

1. Teleport the first qubit to the second one.
2. Do a gate operation involving the second one.

Effectively you have now performed a quantum gate between two distant places where at this time now quantum channel was available

See Gottesman and Chuang, Nature **402** 390-393 (1999), for details.

Qubits with trapped ions

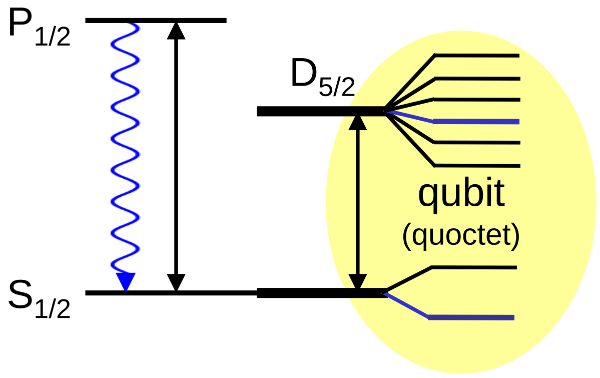
Encoding of quantum information requires long-lived atomic states:

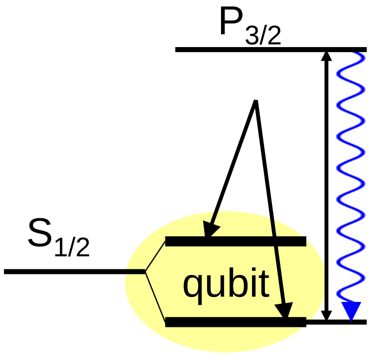
- optical transitions

  Ca$^+$, Sr$^+$, Ba$^+$, Ra$^+$, Yb$^+$, Hg$^+$ etc.

- microwave transitions

  ^{9}Be$^+$, ^{25}Mg$^+$, ^{43}Ca$^+$, ^{87}Sr$^+$, ^{137}Ba$^+$, ^{111}Cd$^+$, ^{171}Yb$^+$

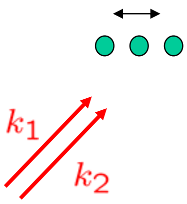




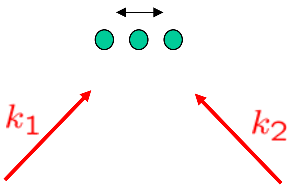
Raman transitions: motional sidebands

$\eta = (\vec{k_2} - \vec{k_1})\vec{n}_{mode}\, x_0$ effective Lamb Dicke parameter

copropagating Raman beams for
excitation of carrier transitions
(no sideband transitions)



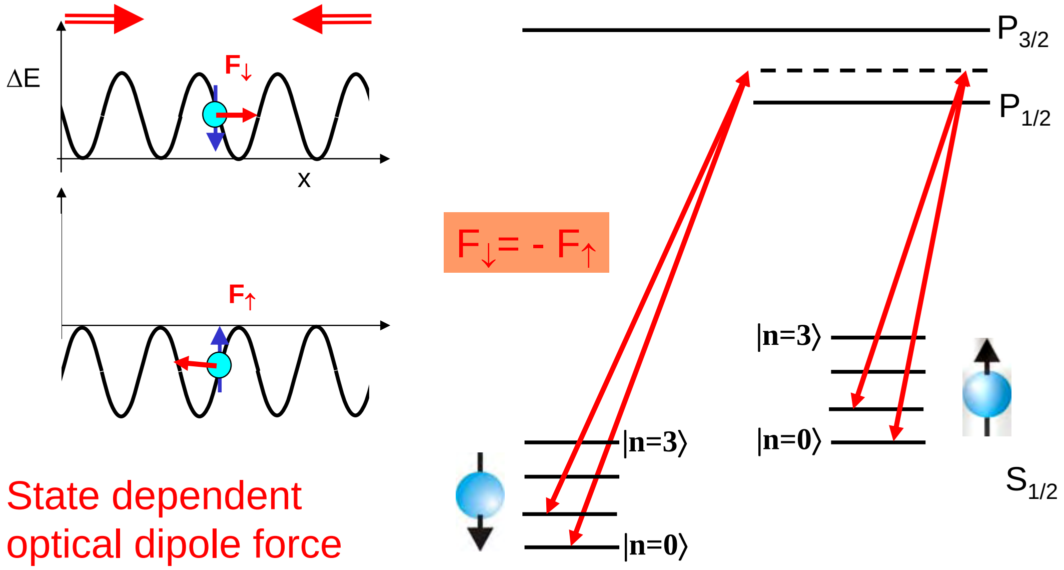
counterpropagating Raman beams for
efficient excitation of sideband transitions
or
beams from different directions



Entangling interactions: controlled phase gate

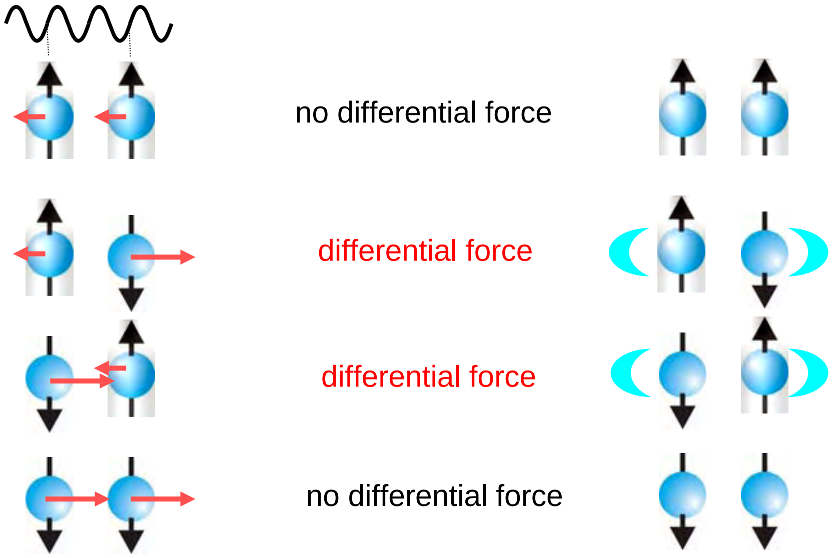
Use Raman beams that couple the motional states (but not internal states)

Raman beams form (moving) standing wave: spatial light shifts



State dependent
optical dipole force

Stretch mode excitation



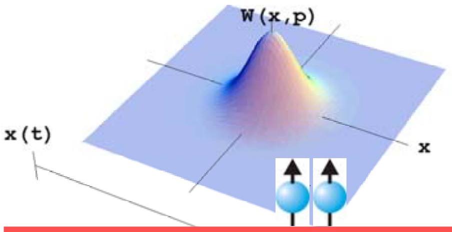
no differential force

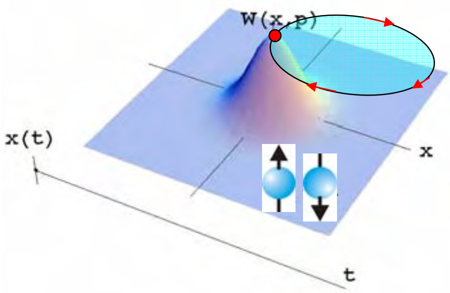
differential force

differential force

no differential force

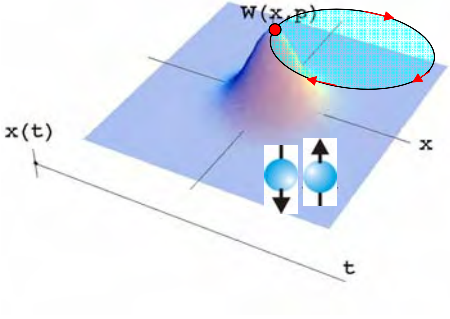
Phase space picture





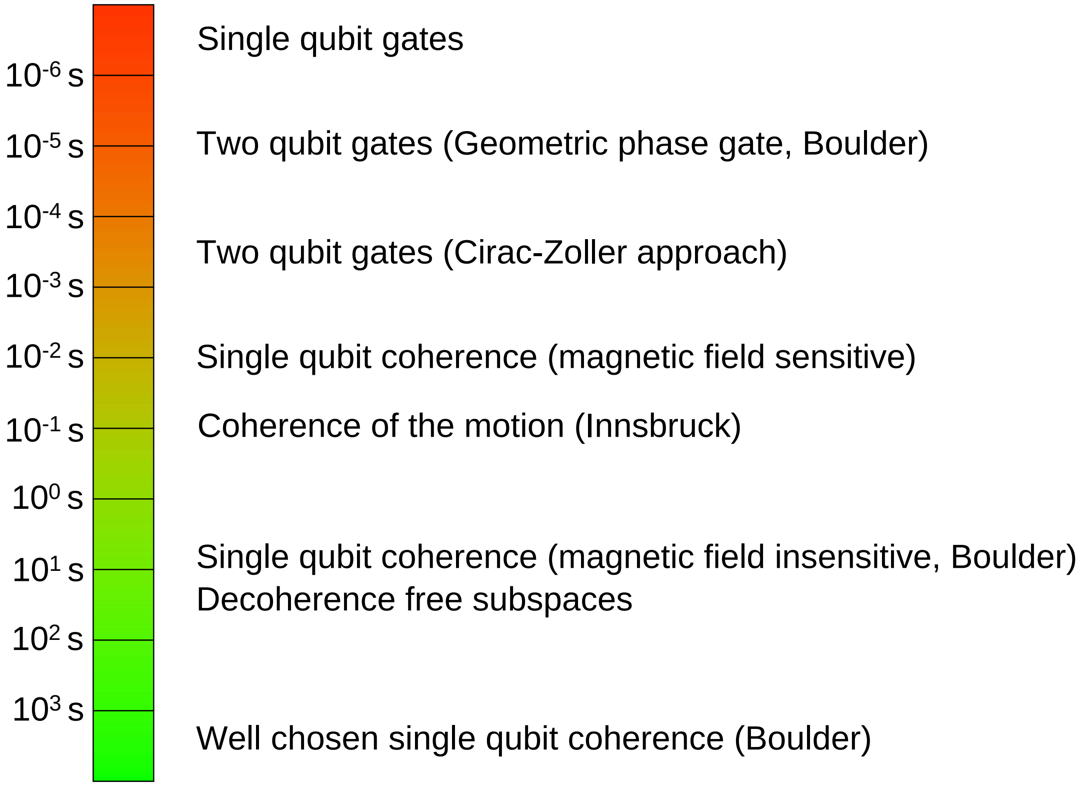
Boulder group :
Gate fidelity: 97%
Gate time: 7 μs (ca. $25/\nu_{COM}$)

D. Leibfried *et al.*, Nature **422**, 414 (2003)
Theory: Milburn, Fortschr. Phys. **48**, 9(2000)
Sørensen&Mølmer



$$
\begin{aligned}
|\downarrow\downarrow\rangle &\rightarrow |\downarrow\downarrow\rangle \\
|\downarrow\uparrow\rangle &\rightarrow e^{i\phi}|\downarrow\uparrow\rangle \\
|\uparrow\downarrow\rangle &\rightarrow e^{i\phi}|\uparrow\downarrow\rangle \\
|\uparrow\uparrow\rangle &\rightarrow |\uparrow\uparrow\rangle
\end{aligned}
$$

Realized time scales in ion trap systems



Single qubit gates

Two qubit gates (Geometric phase gate, Boulder)

Two qubit gates (Cirac-Zoller approach)

Single qubit coherence (magnetic field sensitive)

Coherence of the motion (Innsbruck)

Single qubit coherence (magnetic field insensitive, Boulder)
Decoherence free subspaces

Well chosen single qubit coherence (Boulder)

- Two qubit gate
- Decoherence issues
- Implementation of an algorithm
- Scaling of ion trap quantum computers

Scaling ion trap quantum computers

Easy to have thousands of ions in a trap and to manipulate them individually…

but it is hard to control their interaction!

Scaling of the Cirac-Zoller approach?

Problems :

- Coupling strength between internal and motional states of a N-ion string decreases as

$$\eta \propto \frac{1}{\sqrt{N}}$$

(momentum transfer from photon to ion string becomes more difficult)

-> Gate operation speed slows down

- More vibrational modes increase risk of spurious excitation of unwanted modes
- Distance between neighbouring ions decreases -> addressing more difficult

-> Use flexible trap potentials to split long ion string into smaller segments and perform operations on these smaller strings

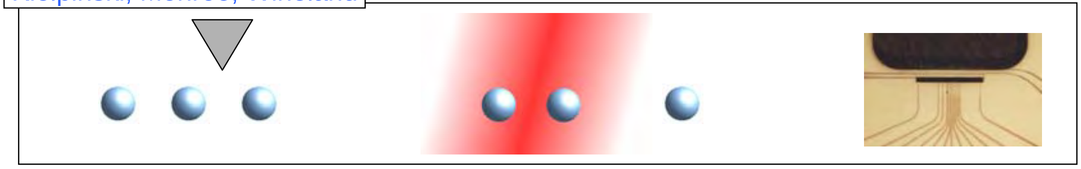
Scaling ion trap quantum computers

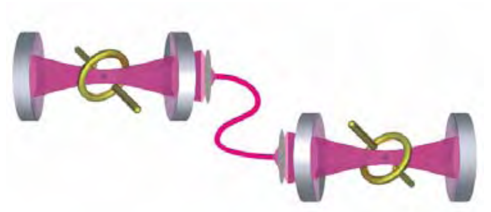
Easy to have thousands of ions in a trap and to manipulate them individually…

but it is hard to control their interaction!

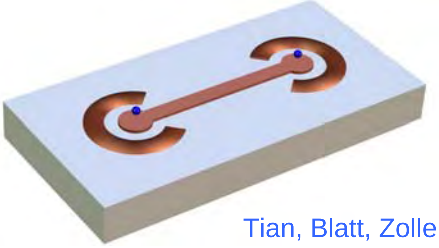
The solutions:

Kielpinski, Monroe, Wineland





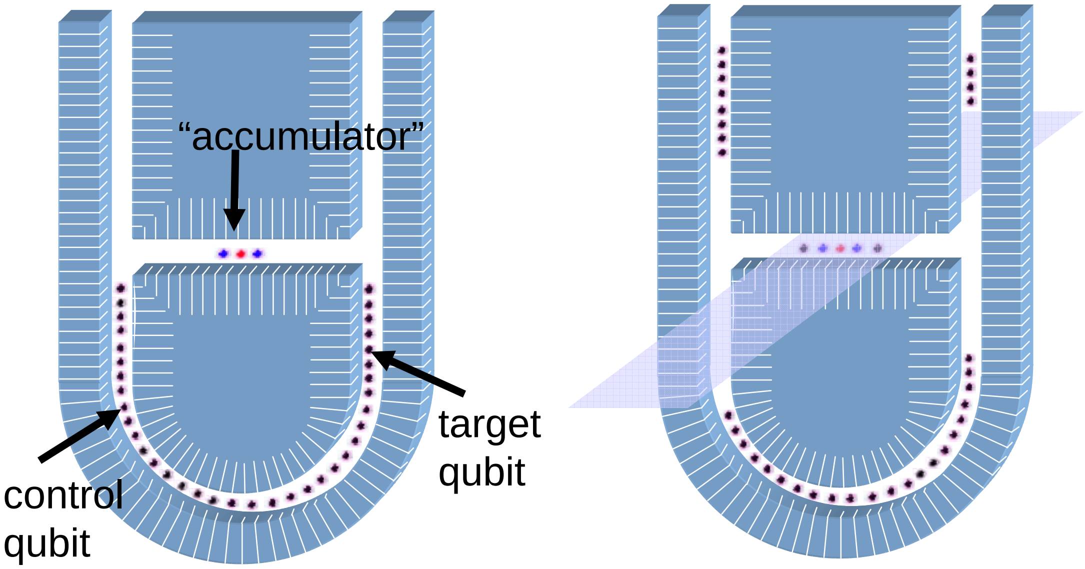
Cirac, Zoller, Kimble, Mabuchi



Tian, Blatt, Zoller

Idea #1: move the ions around

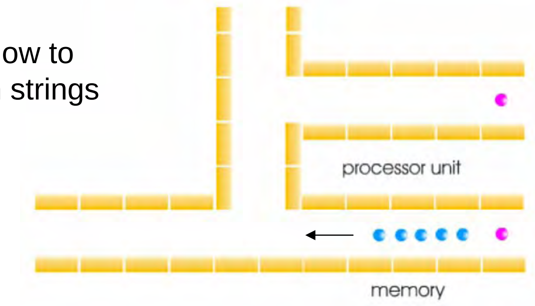
D. Wineland, Boulder, USA
Kieplinski et al, Nature 417, 709 (2002)



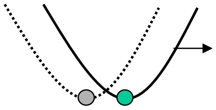
Segmented ion traps as scalable trap architecture

(ideas pioneered by D. Wineland, NIST, and C. Monroe, Univ. Michigan)

Segmented trap electrode allow to transport ions and to split ion strings



State of the art:



Transport of ions

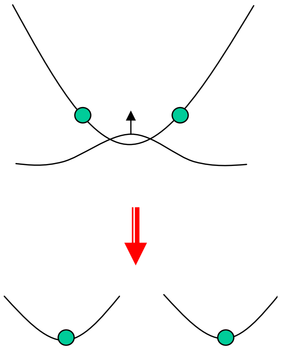
1 mm within 50 μs

no motional heating

Splitting of two-ion crystal

$t_{separation} \approx 200\ \mu s$

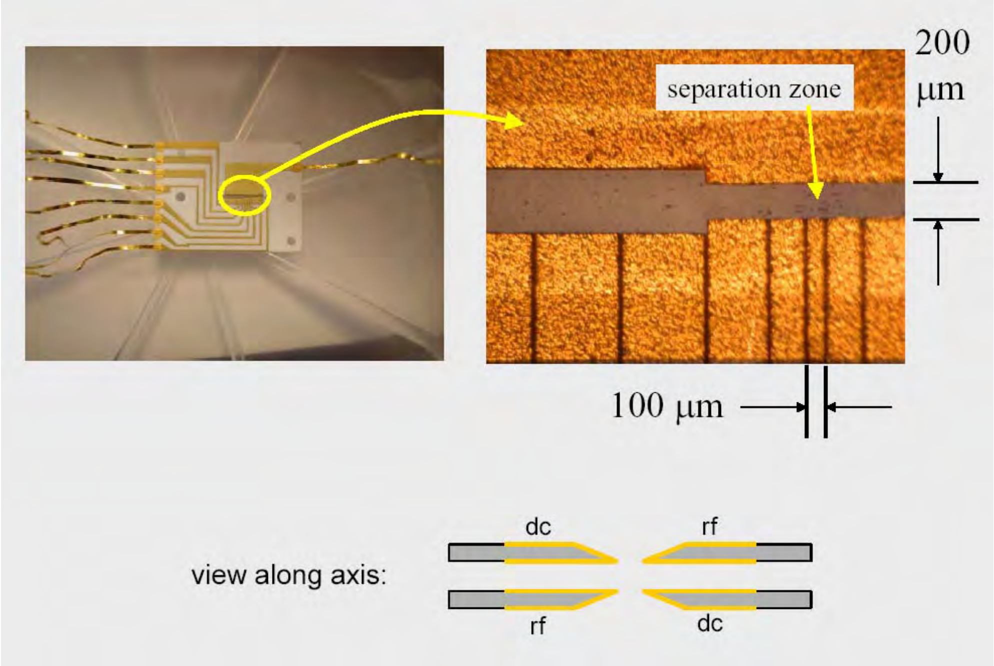
small heating $\Delta n \approx 1$



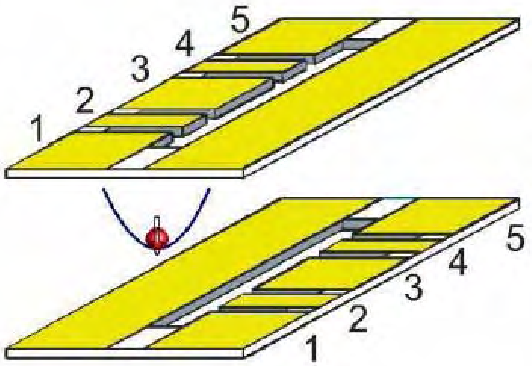
„Architecture for a large-scale ion-trap quantum computer“, D. Kielpinski et al, Nature **417**, 709 (2002)

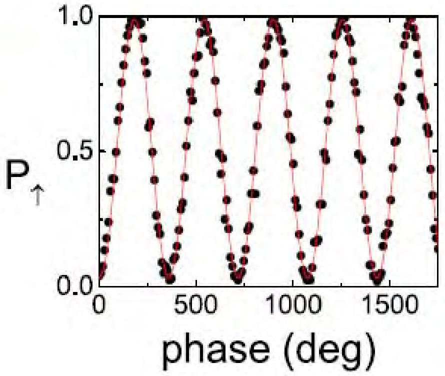
„Transport of quantum states“, M. Rowe et al, quant-ph/0205084

D. Wineland (NIST) : Alumina / gold trap



Coherent transport of quantum information



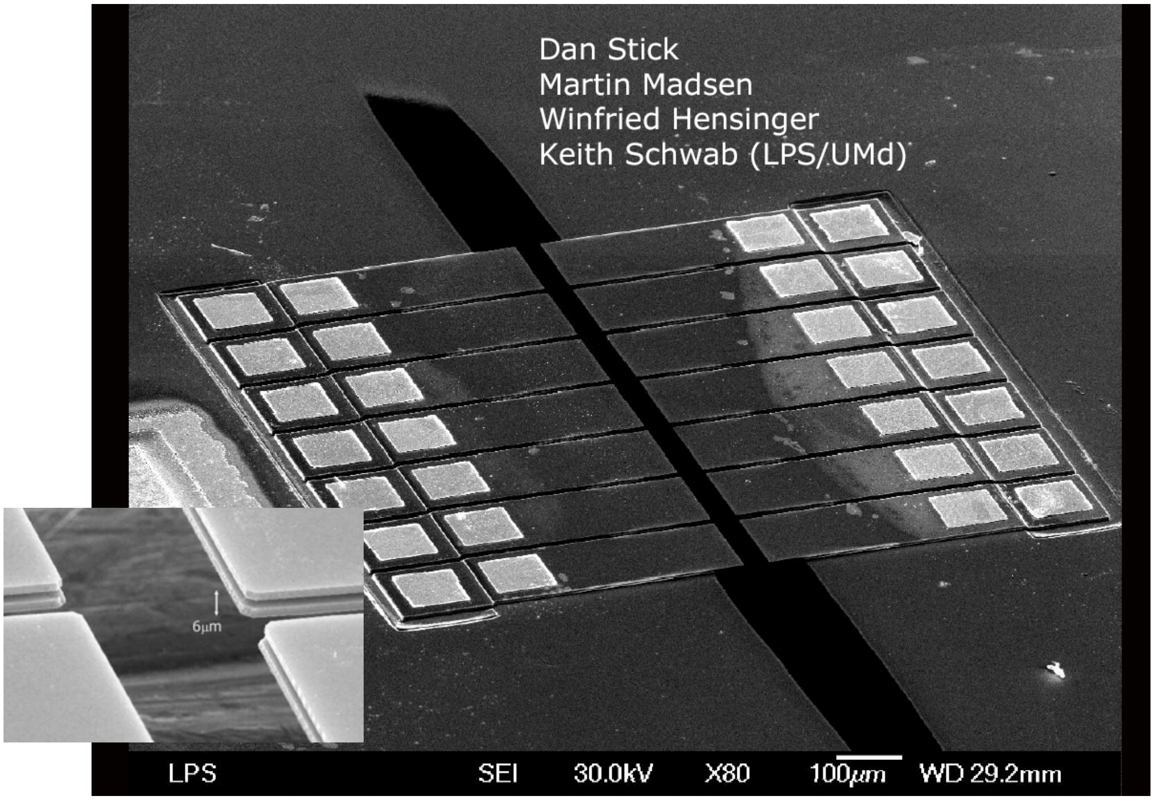


Initial results

- τ(transfer) ≅ 25 μs (motion heating < 1 quantum)
- qubit coherence preserved during transfer (0.5 % measurement accuracy)
- robust (no loss observed from transfer; > 10^6 consecutive transfers typical)
- two ions "split" to separate traps

M.A. Rowe *et al.*, Quant. Info. Compt., **4**, 257 (2002).

C. Monroe (Michigan) : GaAs-GaAlAs trap

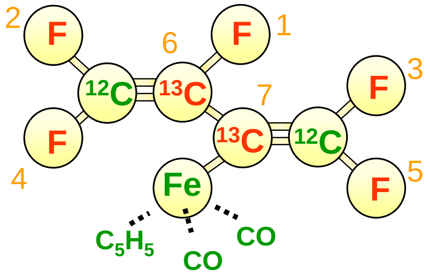


NMR Quantum Computing

Slides courtesy of Lieven Vandersypen (now at TU Delft)
with some annotations by Andreas Wallraff.

How to factor 15 with NMR?

perfluorobutadienyl
iron complex



red nuclei are
qubits: F, ^{13}C

Lieven Vandersypen

Then: IBM Almaden, Stanford University
Now: Kavli Institute of NanoScience, TU Delft

Goals of this tutorial

Survey of NMR quantum computing

- Principles of NMR QC
- Techniques for qubit control
- Example: factoring 15
- State of the art
- Outlook

Perspective on building a quantum computer

- How would *you* factor 15?

NMR largely satisfies the DiVincenzo criteria

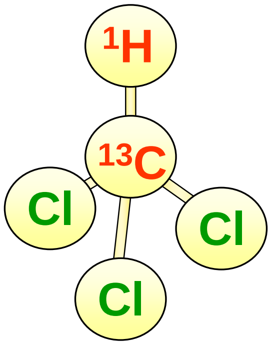
✓ Qubits: nuclear spins ½ in B_0 field (↑ and ↓ as 0 and 1)

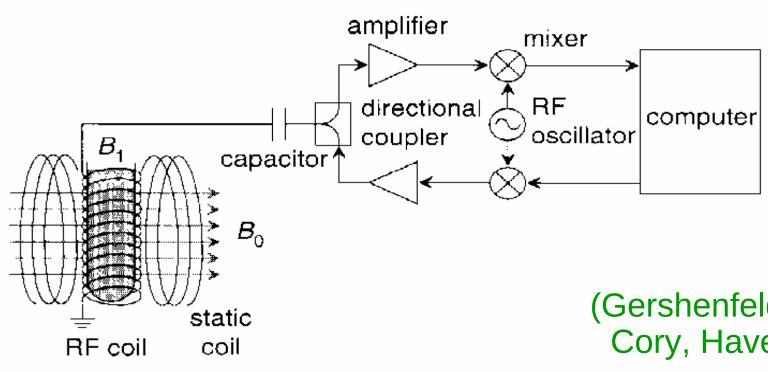
✓ Quantum gates: RF pulses and delay times

(✓) Input: Boltzman distribution (room temperature)

✓ Readout: detect spin states with RF coil

✓ Coherence times: easily several seconds



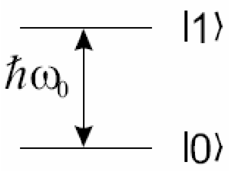


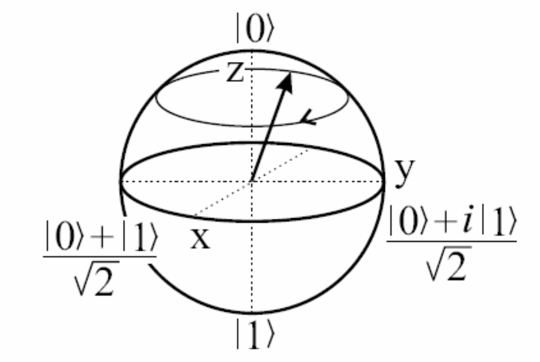
(Gershenfeld & Chuang 1997,
Cory, Havel & Fahmi 1997)

Nuclear spin Hamiltonian

Single spin

$$\mathcal{H}_0 = -\hbar\gamma B_0\, I_z = -\hbar\,\omega_0\; I_z = \begin{bmatrix} -\hbar\omega_0/2 & 0 \\ 0 & \hbar\omega_0/2 \end{bmatrix}$$





angular momentum: $\vec{L} = \hbar\,\vec{I}$

magnetic moment: $\vec{M} = \gamma\,\hbar\,\vec{I}$

energy: $H_0 = -\,\vec{M}\cdot\vec{B}_0$

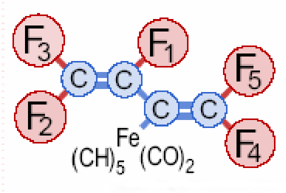
gyromagnetic (g-)factor: γ

Nuclear spin Hamiltonian

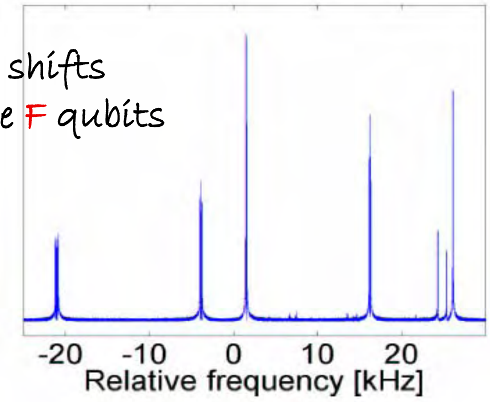
Multiple spins

without qubit/qubit coupling

$$\mathcal{H}_0 = -\sum_{i=1}^{n} \hbar\,(1-\tilde{\sigma}_i)\gamma_i B_0\; I_z^i = -\sum_{i=1}^{n} \hbar\,\omega_0^i\; I_z^i$$



chemical shifts of the five F qubits



	MHz	
^{1}H	500	~ 25 mK
^{13}C	126	
^{15}N	-51	
^{19}F	470	
^{31}P	202	

(at 11.7 Tesla)

qubit level separation

Hamiltonian with RF field

single-qubit rotations

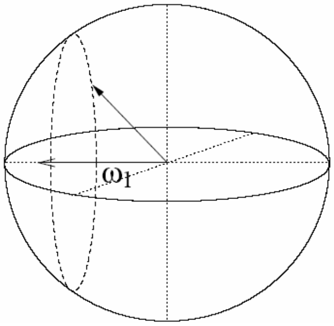
$$\mathcal{H} = -\hbar\,\omega_0\, \overset{\sigma_z}{I_z} - \hbar\omega_1 \left[\cos(\omega_{rf}t+\phi)\overset{\sigma_x}{I_x} + \sin(\omega_{rf}t+\phi)\overset{\sigma_y}{I_y}\right]$$

$$|\psi\rangle^{rot} = \exp(-i\omega_{rf}tI_z)|\psi\rangle$$

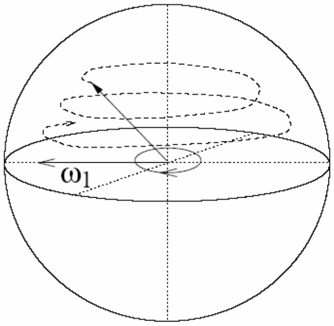
$$\mathcal{H}^{rot} = -\hbar\,(\omega_0-\omega_{rf})\; I_z - \hbar\omega_1 \left[\cos\phi\, I_x + \sin\phi\, I_y\right]$$

rotating wave approximation

typical strength I_x, I_y : up to 100 kHz



Rotating frame



Lab frame

Nuclear spin Hamiltonian

Coupled spins

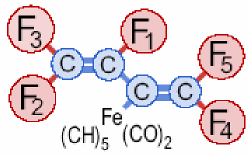
J>0: antiferro mag.

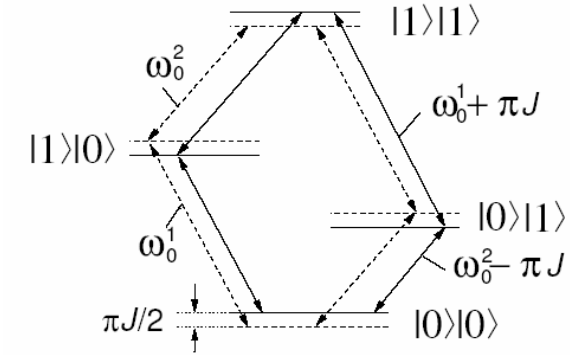
J<0: ferro-mag.

coupling term

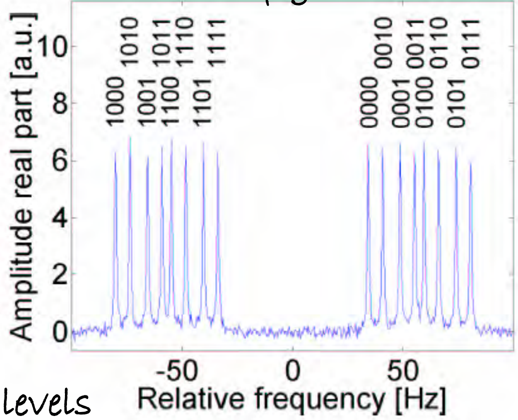
$$\mathcal{H}_J = \hbar\sum_{i<j}^{n} 2\pi J_{ij} I_z^i I_z^j$$

Typical values: *J* up to few 100 Hz

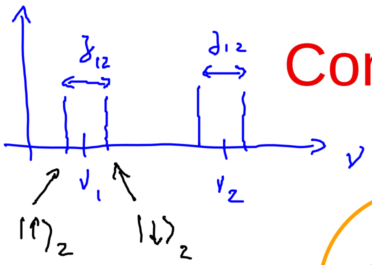




16 configurations

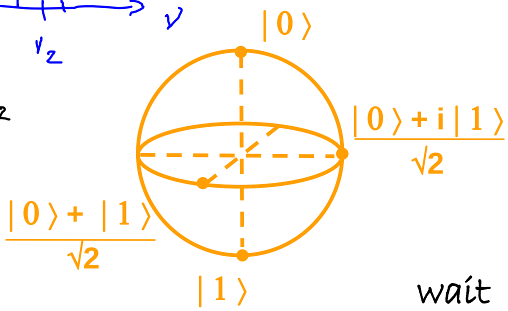


solid (dashed) lines are (un)coupled levels



Controlled-NOT in NMR

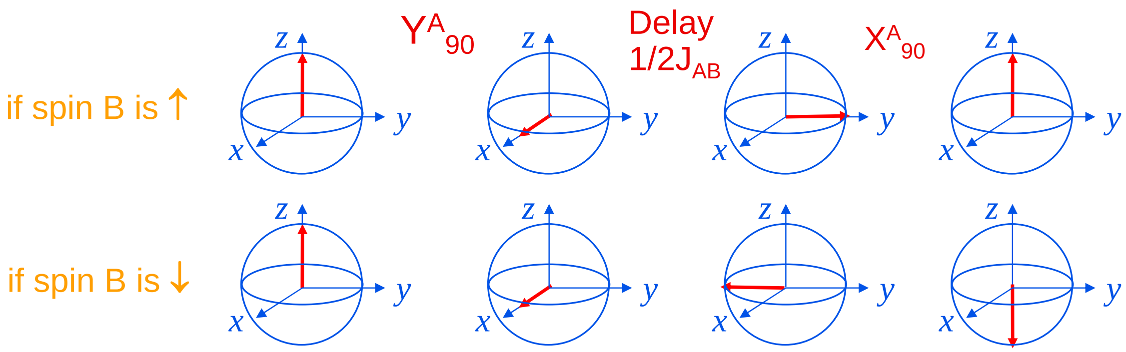
A target bit
B control bit



wait

Before		After	
A	B	A	B
↑	↑	↑	↑
↑	↓	↓	↓
↓	↑	↓	↑
↓	↓	↑	↓

" flip A if B ↓"

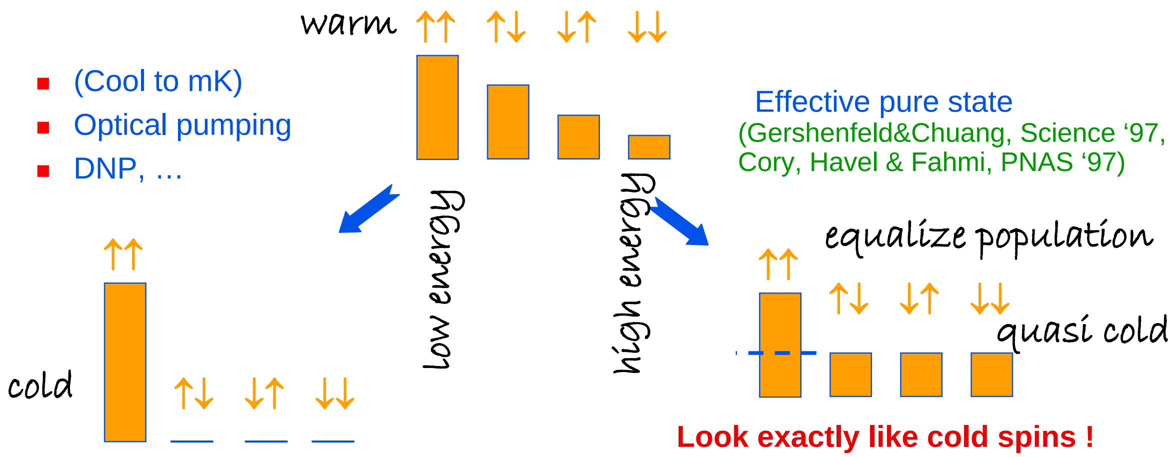


different rotation direction depending on control bit

→ time

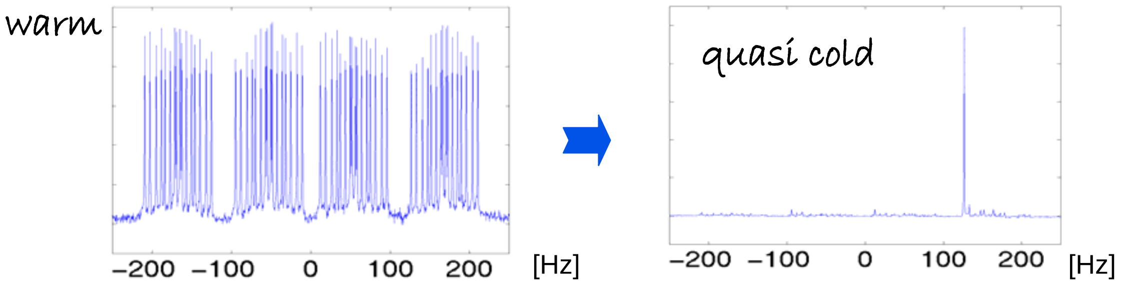
Making room temperature spins look cold

- (Cool to mK)
- Optical pumping
- DNP, …



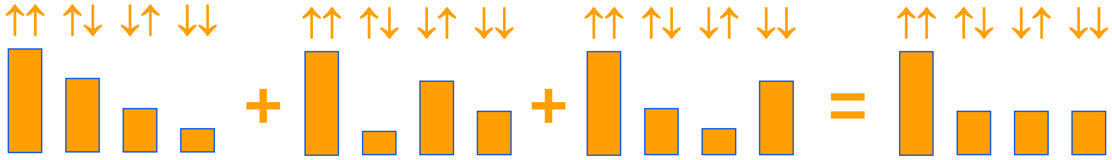
Effective pure state
(Gershenfeld&Chuang, Science '97, Cory, Havel & Fahmi, PNAS '97)

Look exactly like cold spins !



Effective pure state preparation

(1) Add up 2^N-1 experiments (Knill,Chuang,Laflamme, PRA 1998)



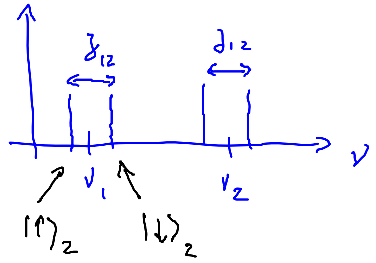
Later ≈ (2^N - 1) / N experiments (Vandersypen *et al.*, PRL 2000)

prepare equal population (on average) and look at deviations fromequilibrium.

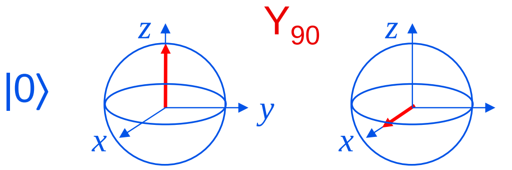
(2) Work in subspace (Gershenfeld&Chuang, Science 1997)

compute with qubit states that have the same energy and thus the same population.

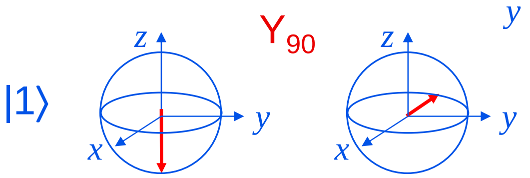
Read-out in NMR



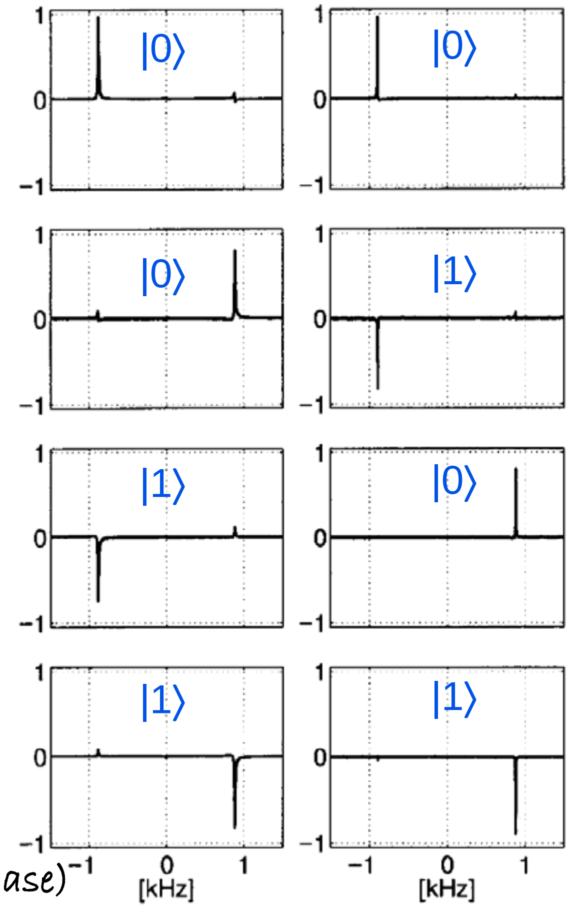
Phase sensitive detection



positive signal for |0> (in phase)



negative signal for |1> (out of phase)



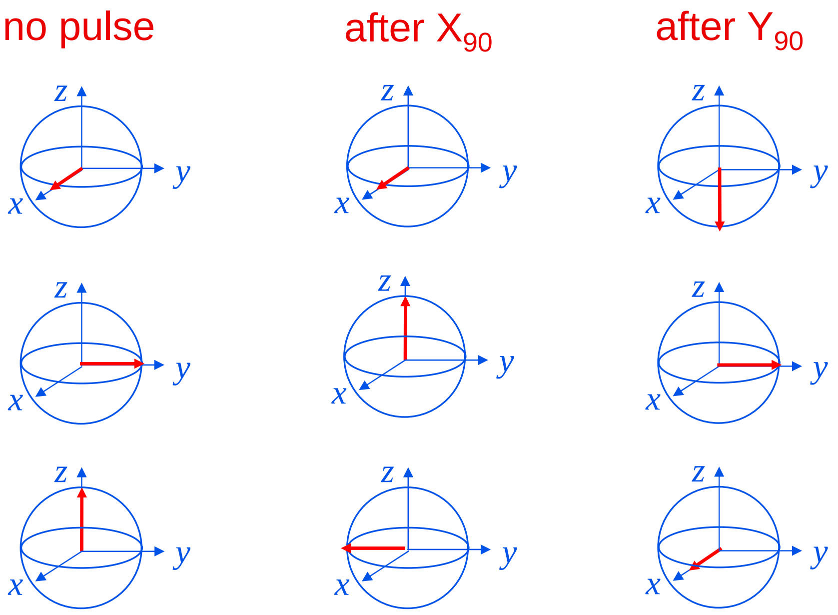
Measurements of single systems versus ensemble measurements

quantum state	|00⟩	|00⟩ + |11⟩	
single-shot bitwise	|0⟩ and |0⟩	each bit |0⟩ *or* |1⟩	
single-shot “word”wise	|00⟩	|00⟩ *or* |11⟩	**QC**
bitwise average	|0⟩ and |0⟩	each bit average of |0⟩ and |1⟩	**NMR**
“word”wise average	|00⟩	average of |00⟩ and |11⟩	

adapt algorithms if use ensemble

Quantum state tomography

Look at qubits from different angles



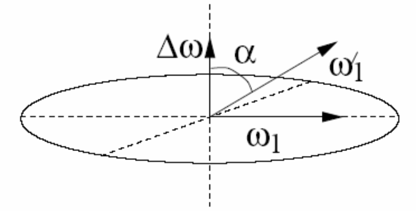
Outline

Survey of NMR quantum computing

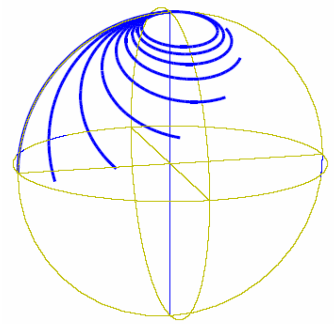
- Principles of NMR QC
- → Techniques for qubit control
- Example: factoring 15
- State of the art
- Outlook

Off-resonance pulses and spin-selectivity

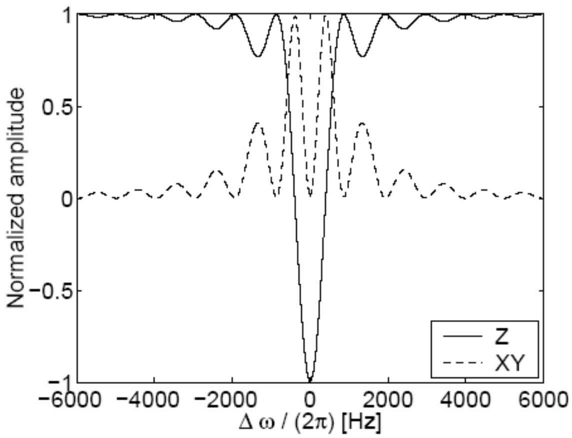
$$\mathcal{H}^{rot} = -\hbar\,(\omega_0 - \omega_{rf})\; I_z - \hbar\omega_1 \left[\cos\phi\, I_x + \sin\phi\, I_y\right]$$



off-resonant pulses induce eff. $\boldsymbol{\sigma}_z$ rotation in addition to $\boldsymbol{\sigma}_{x,y}$



spectral content of a square pulse

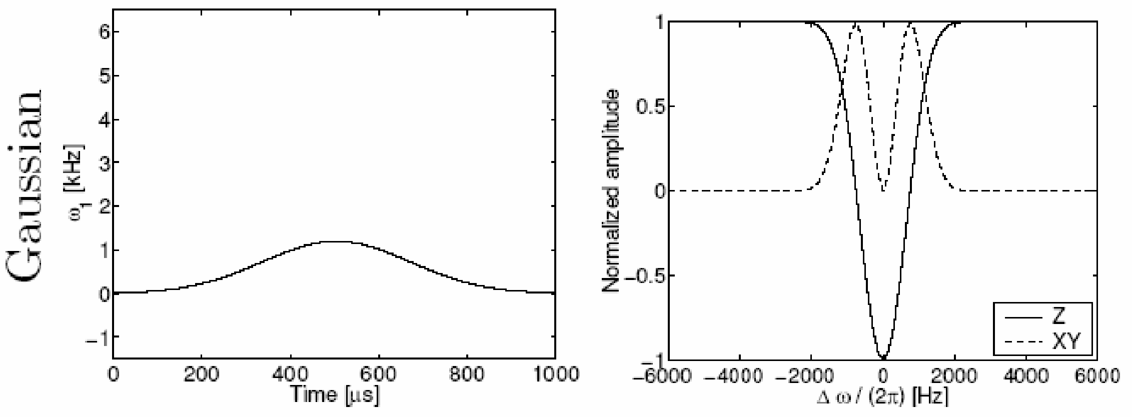


may induce transitions in other qubits

Pulse shaping for improved spin-selectivity

gaussian pulse profile

gaussian spectrum



less cross-talk

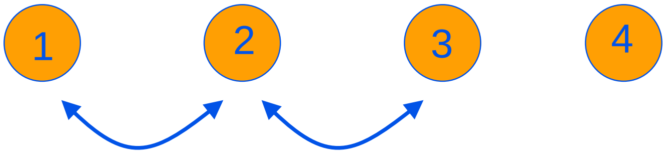
Missing coupling terms: Swap

How to couple distant qubits with only nearest neighbor physical couplings?

Missing couplings: swap states along qubit network

$\mathrm{SWAP}_{12} = \mathrm{CNOT}_{12}\ \mathrm{CNOT}_{21}\ \mathrm{CNOT}_{12}$

as discussed in exercise class

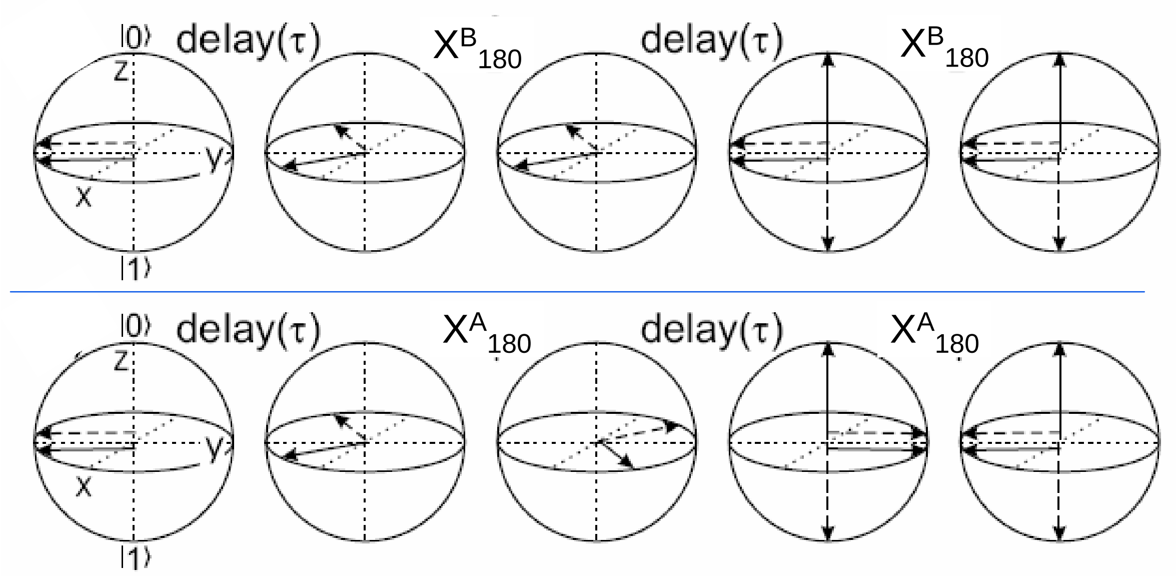


“only” a linear overhead ...

Undesired couplings: refocus

remove effect of coupling *during delay times*

opt. 1: act on qubit B



opt. 2: act on qubit A

- There exist efficient extensions for arbitrary coupling networks
- Refocusing can also be used to remove unwanted Zeeman terms

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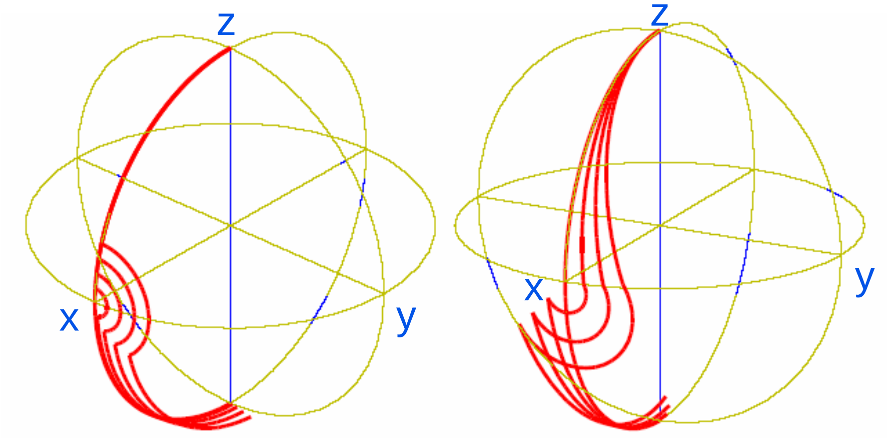
Composite pulses

Example: $Y_{90}X_{180}Y_{90}$

corrects for
under/over-rotation

corrects for
off-resonance



However: doesn't work for arbitrary input state
But: there exist composite pulses that work for all input states

Molecule selection

A quantum computer is a *known* molecule.
Its desired properties are:

- spins 1/2 (^{1}H, ^{13}C, ^{19}F, ^{15}N, ...)
- long T_1's and T_2's
- heteronuclear, or large chemical shifts
- good J-coupling network (clock-speed)

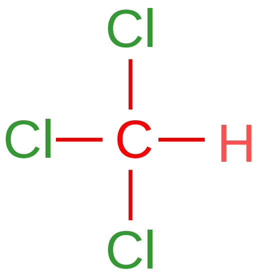
- stable, available, soluble, ...

required to make
spins of same
type addressable

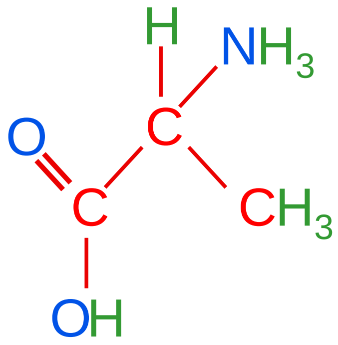
Quantum computer molecules (1)

red nuclei are used
as qubits:

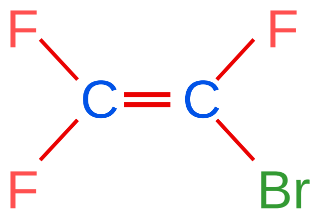
Grover / Deutsch-Jozsa



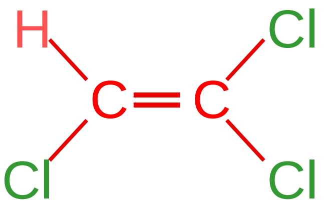
Q. Error correction



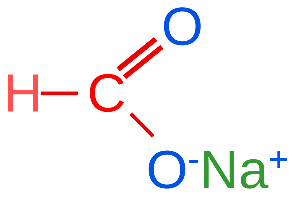
Logical labeling / Grover



Teleportation

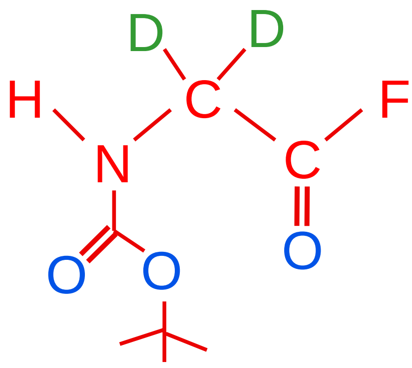


Q. Error Detection

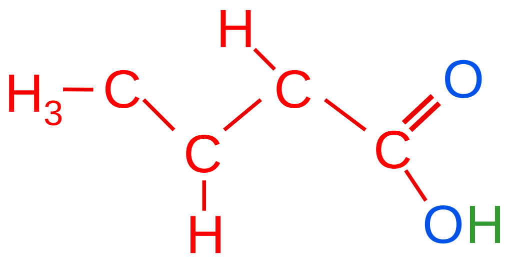


Quantum computer molecules (2)

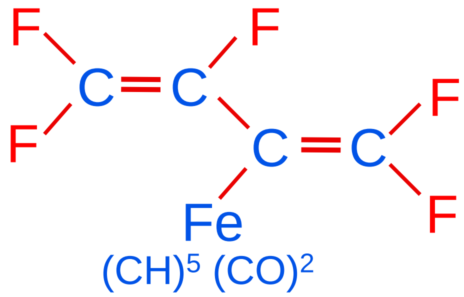
Deutsch-Jozsa



7-spin coherence



Order-finding



Outline

Survey of NMR quantum computing

- Principles of NMR QC
- Techniques for qubit control
- ➡ Example: factoring 15
- State of the art
- Outlook

Quantum Factoring

Find the prime factors of N: chose a and find order r.

$f(x) = a^x \bmod N$

coprime with N

composite number

Results from number theory:

- f is periodic in x (period r)
- $\gcd(a^{r/2} \pm 1, N)$ is a factor of N

Quantum factoring: find r

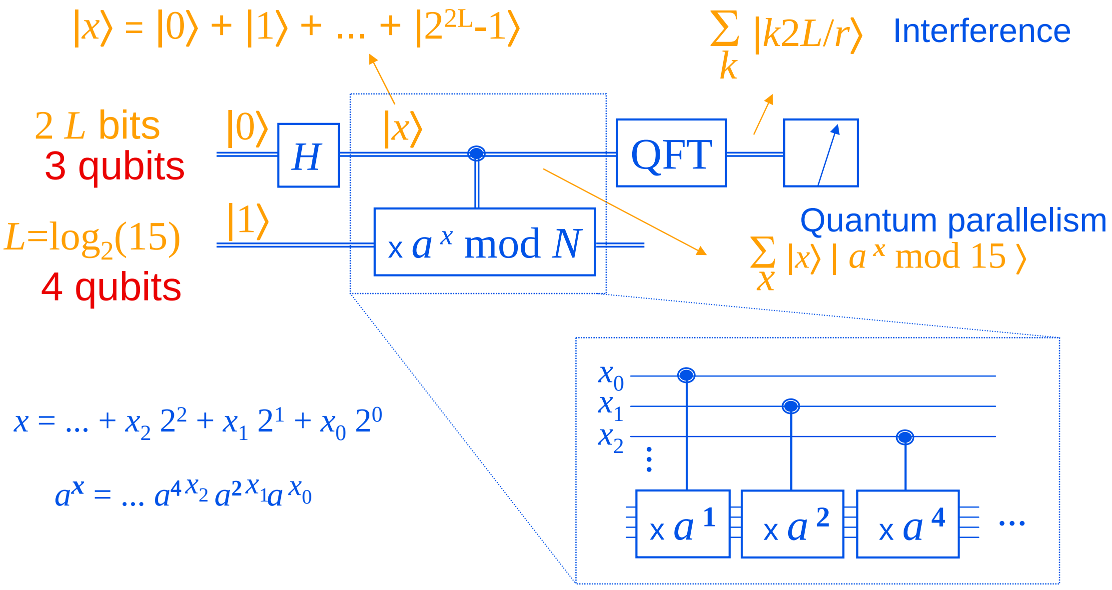
Complexity of factoring numbers of length L:

Quantum: ~ L^3 P. Shor (1994)

Classically: ~ $e^{L/3}$

Widely used crypto systems (RSA) would become insecure.

Factoring 15 - schematic



$x = \ldots + x_2\, 2^2 + x_1\, 2^1 + x_0\, 2^0$

$a^x = \ldots\, a^{4x_2}\, a^{2x_1} a^{x_0}$

a = 4, 11 ⇒ a^2 mod 15 = 1 ⇒ "easy" case

a = 2, 7, 8, 13 ⇒ a^4 mod 15 = 1 ⇒ "hard" case

a = 14 ⇒ fails

Quantum Fourier transform and the FFT

FFT

[1 1 1 1 1 1 1 1]	[1 . . . .]
[1 . 1 . 1 . 1 .]	[1 . . . 1 . . .]
[1 . . . 1 . . .]	[1 . 1 . 1 . 1 .]
[1 . . . .]	[1 1 1 1 1 1 1 1]
[1 . . . 1 . . .]	[1 . 1 . 1 . 1 .]
[. 1 . . . 1 . .]	[1 . -i . -1 . i .]
[. . 1 . . . 1 .]	[1 . -1 . 1 . -1 .]
[. . . 1 . . . 1]	[1 . i . -1 . -i .]

The FFT (and QFT)
- Inverts the period
- Removes the off-set

$$|\psi_3\rangle = |0\rangle\,|0\rangle + |1\rangle\,|2\rangle + |2\rangle\,|0\rangle + |3\rangle\,|2\rangle + |4\rangle\,|0\rangle + |5\rangle\,|2\rangle + |6\rangle\,|0\rangle + |7\rangle\,|2\rangle$$

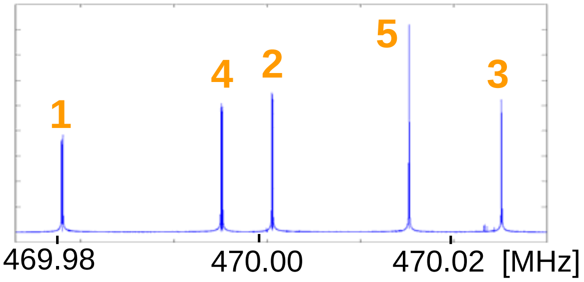
$$= (\,|0\rangle + |2\rangle + |4\rangle + |6\rangle\,)\,|0\rangle + (\,|1\rangle + |3\rangle + |5\rangle + |7\rangle\,)\,|2\rangle$$ hard

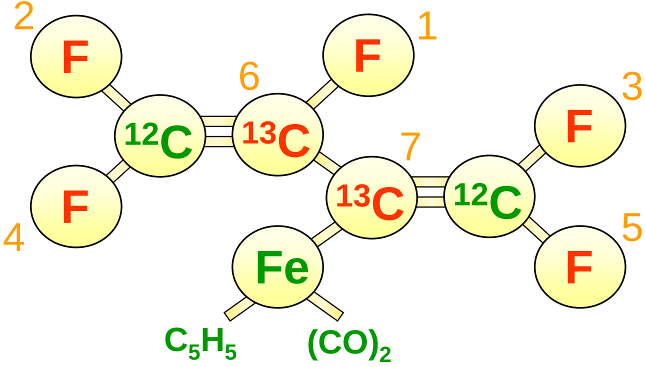
$$|\psi_4\rangle = (\,|0\rangle + |4\rangle\,)\,|0\rangle + (\,|0\rangle - |4\rangle\,)\,|2\rangle$$ easy

---

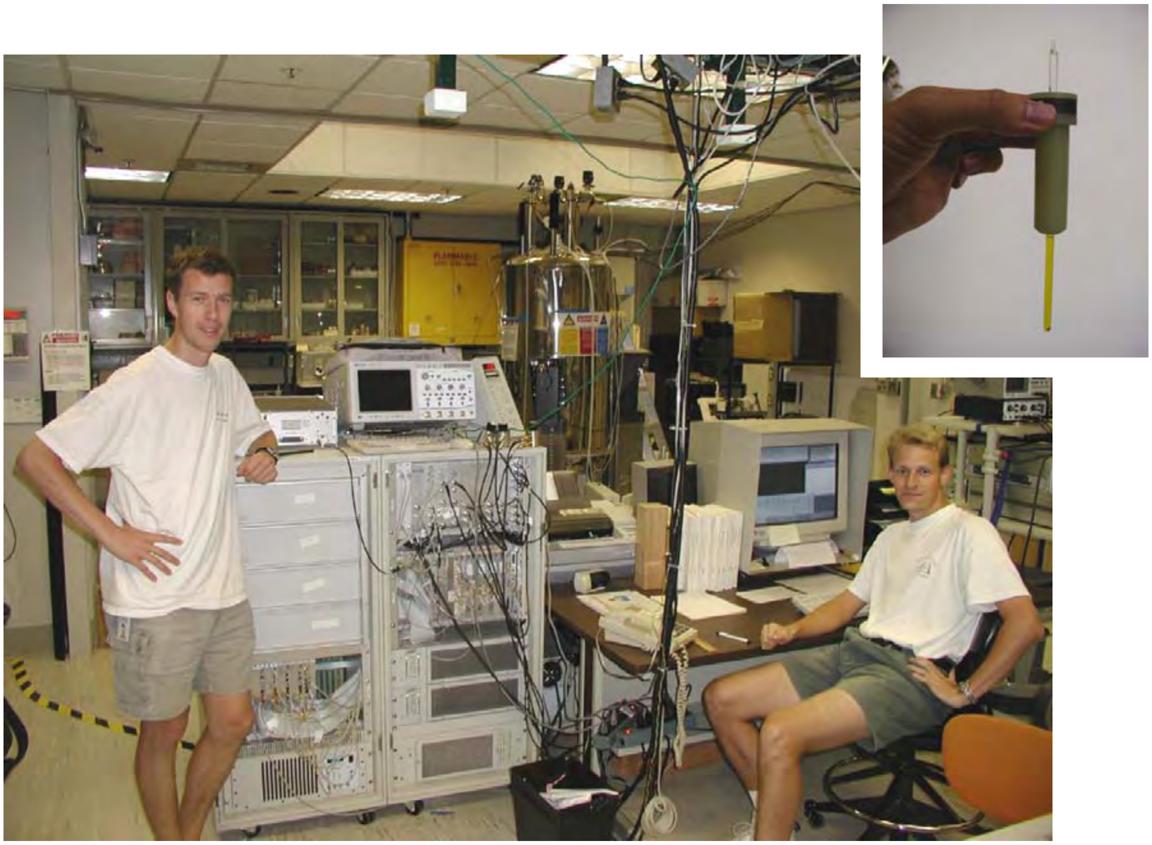
Experimental approach

- 11.7 Tesla Oxford superconducting magnet; room temperature bore
- 4-channel Varian spectrometer; need to address and keep track of 7 spins
  - phase ramped pulses
  - software reference frame
- Shaped pulses
- Compensate for cross-talk
- Unwind coupling during pulses



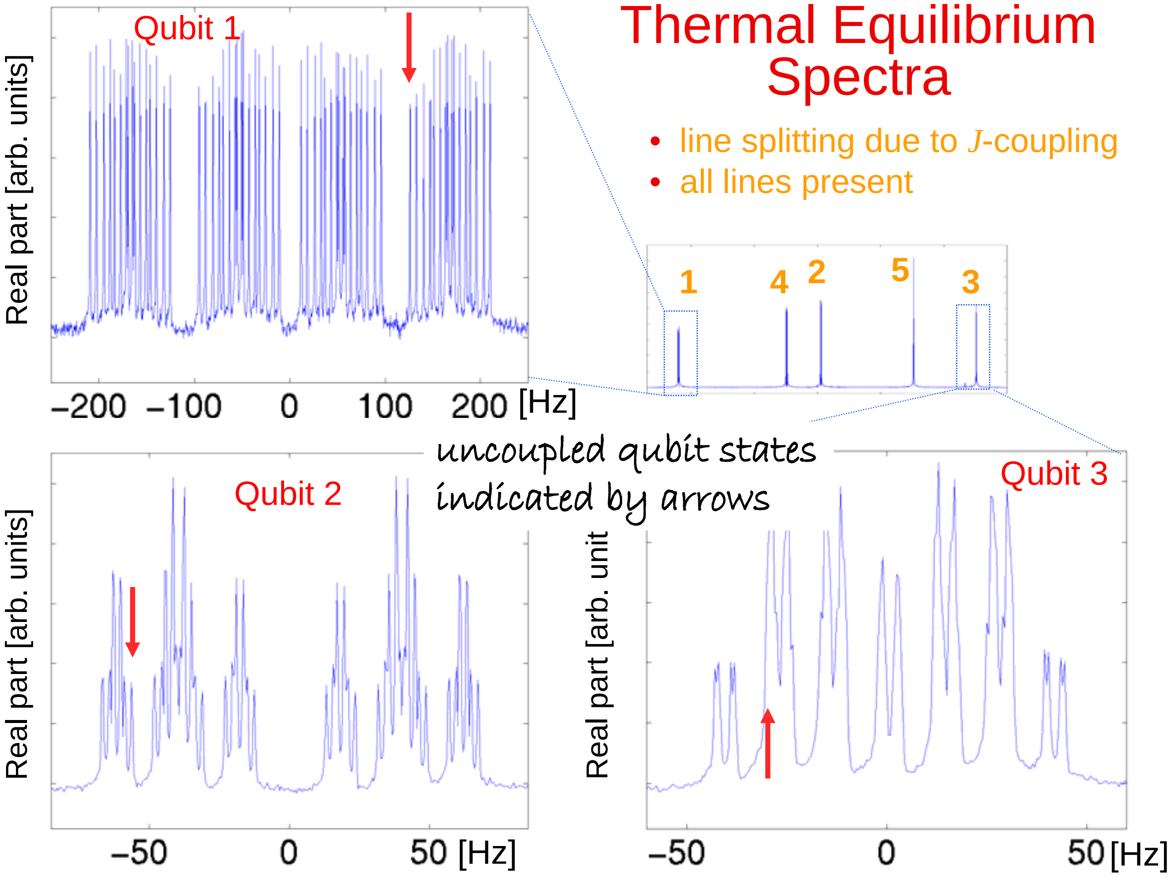


- Larmor frequencies
  - 470 MHz for ^{19}F ~25 mK
  - 125 MHz for ^{13}C
- *J* - couplings: 2 - 220 Hz
- coherence times: 1.3 - 2 s

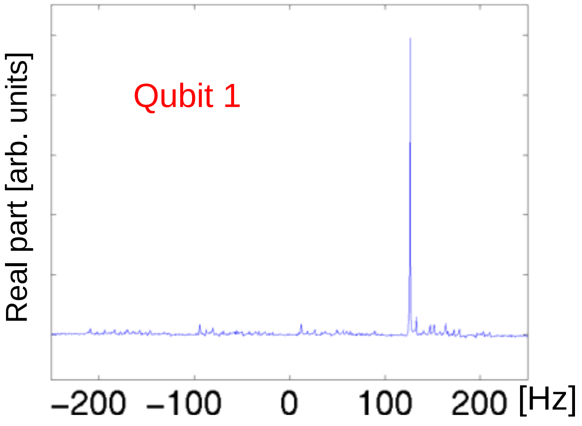


Thermal Equilibrium Spectra

- line splitting due to *J*-coupling
- all lines present



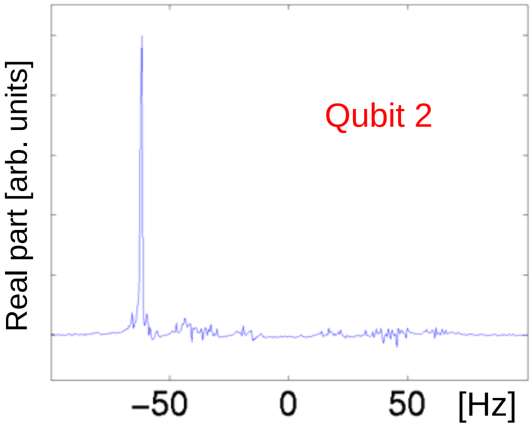
uncoupled qubit states
indicated by arrows

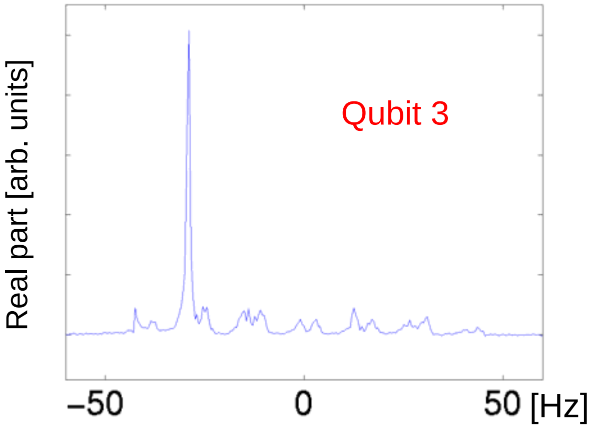


Spectra after state initialization

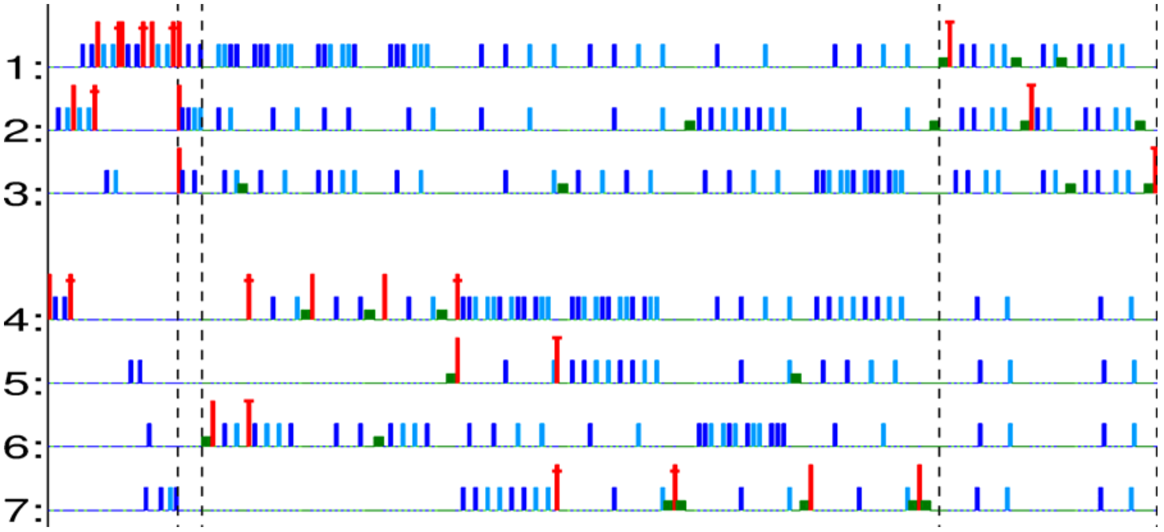
- only the |00 … 0 ⟩ line remains
- the other lines are averaged away by adding up multiple experiments

RT spins appear cold!





Pulse sequence (*a*=7)



π/2 X- or Y-rotations (H and gates)

π X-rotations (refocusing)

Z - rotations

> 300 pulses, ≈ 720 ms

“Easy” case (a=11)

321

|000⟩ **0**

|100⟩ **4**

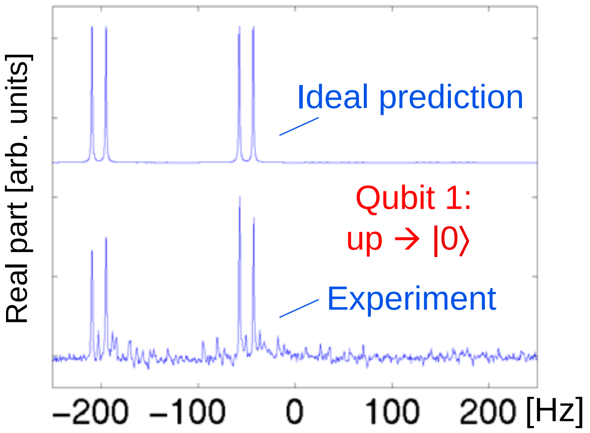
→ 8 / r = 4

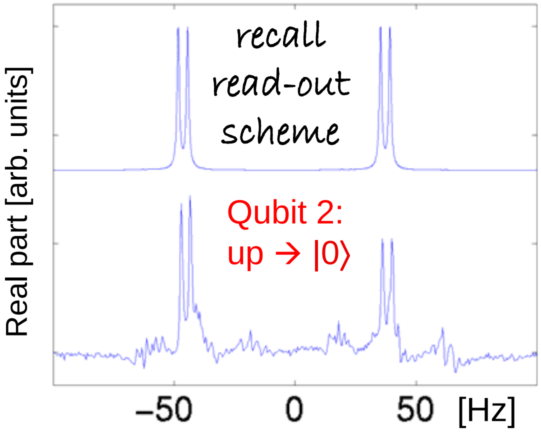
r = 2 period

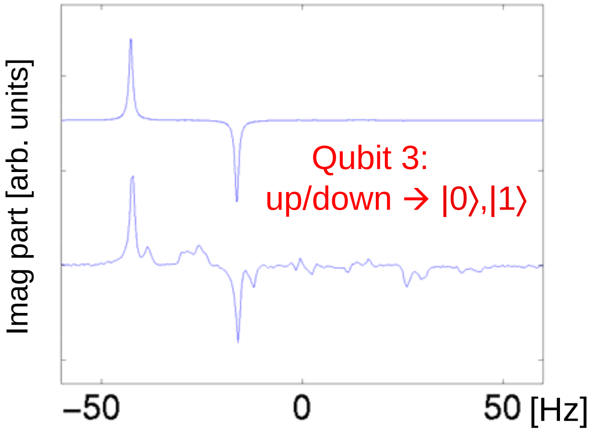
$\gcd(11^{2/2} - 1, 15) = 5$

$\gcd(11^{2/2} + 1, 15) = 3$

15 = 3 x 5







“Hard” case (a=7)

|000⟩ **0**

|010⟩ **2**

|100⟩ **4**

|110⟩ **6**

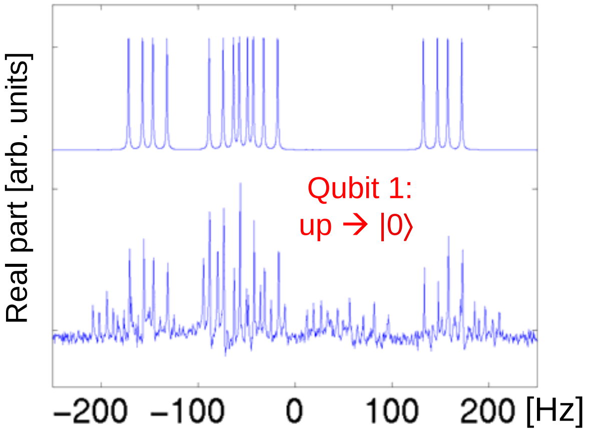
→ 8 / r = 2

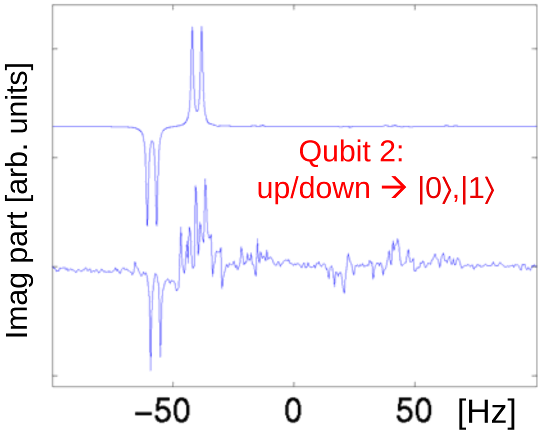
r = 4 period

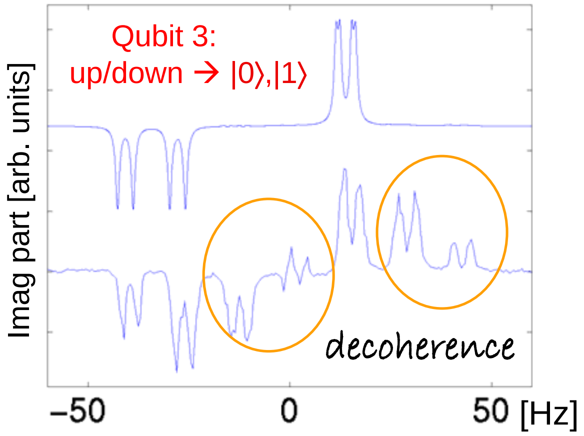
$\gcd(7^{4/2} - 1, 15) = 3$

$\gcd(7^{4/2} + 1, 15) = 5$

15 ≅ 3 x 5







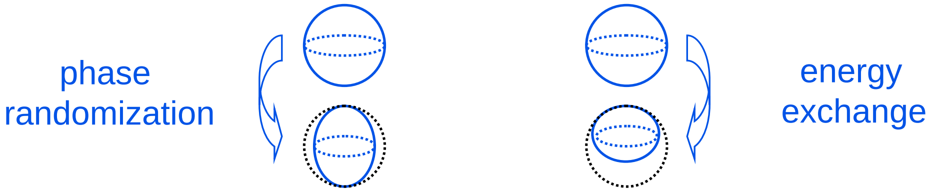
Model quantum noise (decoherence)

Spins interact with the environment

⬇

Decoherence

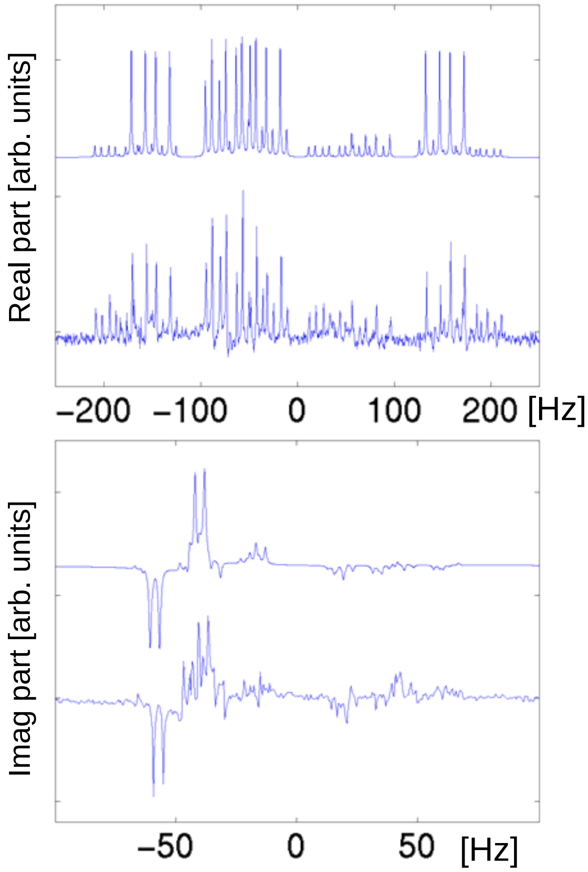
The decoherence model for 1 nuclear spin is well-described.



phase
randomization

energy
exchange

We created a workable decoherence model for 7 coupled spins.
The model is parameter free.

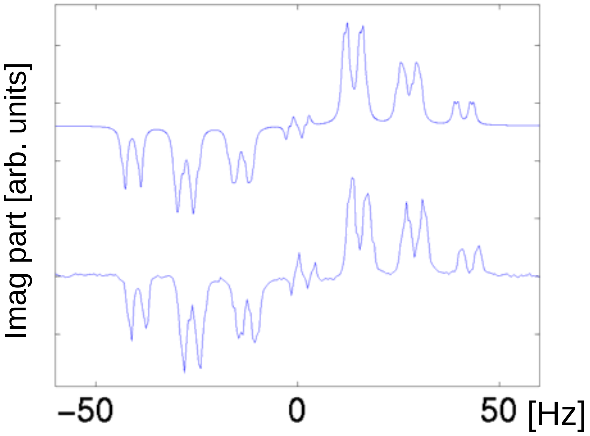


Simulation of decoherence (1)

fundamental limit

hard case

decoherence can be
understood and modeled

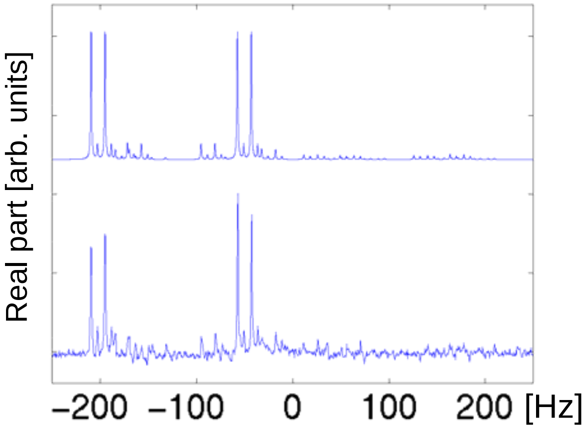


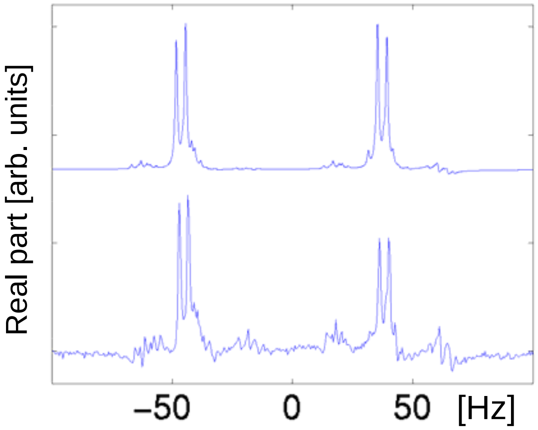
Simulation of decoherence (2)

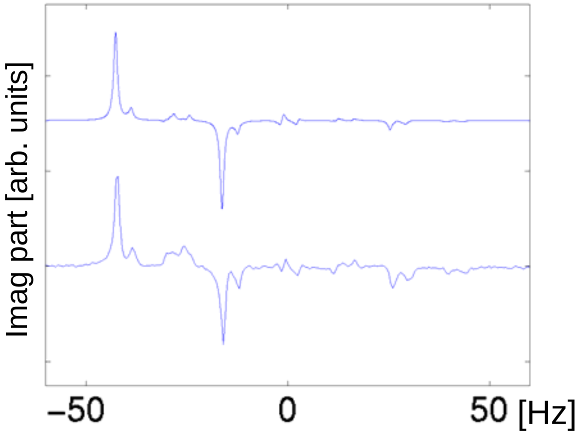
fundamental limit

easy case

decoherence can be understood and modeled





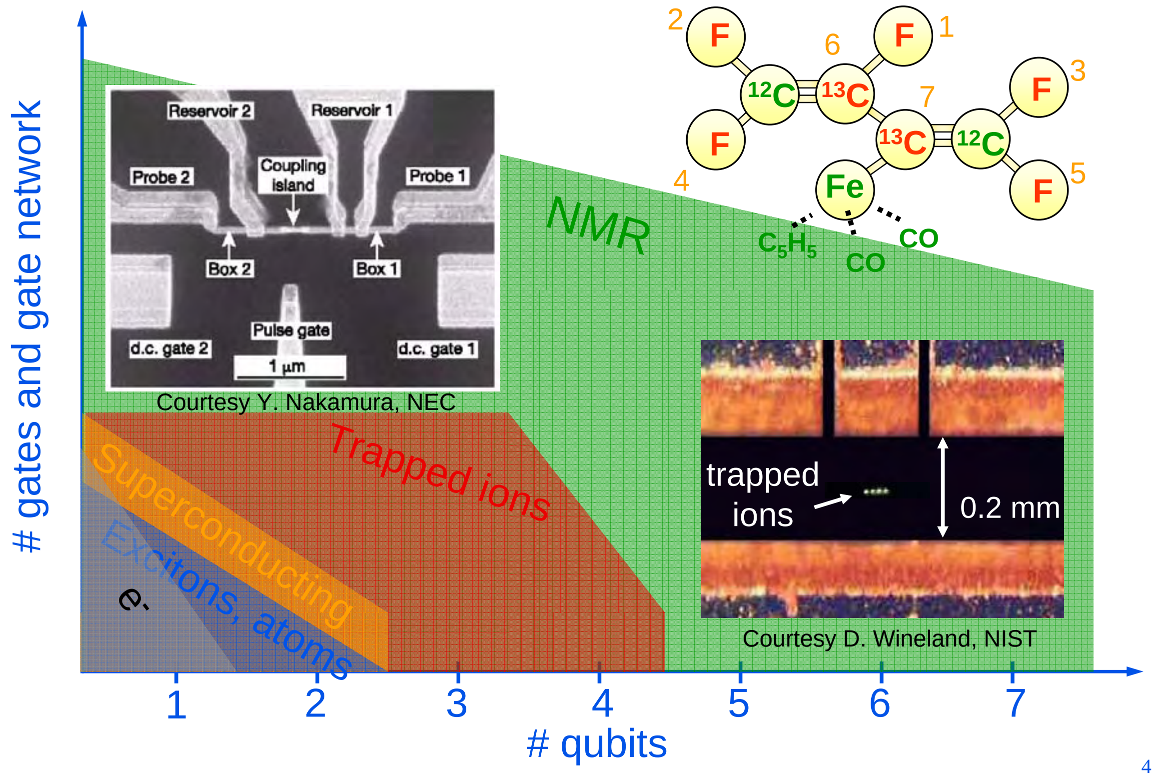


Outline

Survey of NMR quantum computing

- Principles of NMR QC
- Techniques for qubit control
- Example: factoring 15
- → State of the art
- Outlook

State of the art in various qubit systems



Courtesy Y. Nakamura, NEC

Courtesy D. Wineland, NIST

The good news

- Quantum computations have been demonstrated in the lab
- A high degree of control was reached, permitting hundreds of operations in sequence
- A variety of tools were developed for accurate unitary control over multiple coupled qubits
  - ⇨ *useful in other quantum computer realizations*
- Spins are natural, attractive qubits

Scaling

We do not know how to scale liquid NMR QC

Main obstacles:

- Signal after initialization ~ 1 / 2^n [at least in practice]
- Coherence time typically goes down with molecule size
- We have not yet reached the accuracy threshold ...

Main sources of errors in NMR QC

Early on (heteronuclear molecules)

inhomogeneity RF field

Later (homonuclear molecules)

*J c*oupling during RF pulses

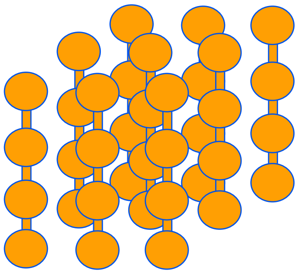
Finally

decoherence

Solid-state NMR ?

molecules in
solid matrix

Cory et al

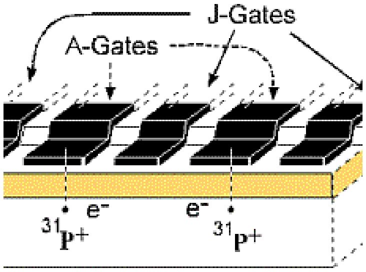


Yamaguchi & Yamamoto, 2000

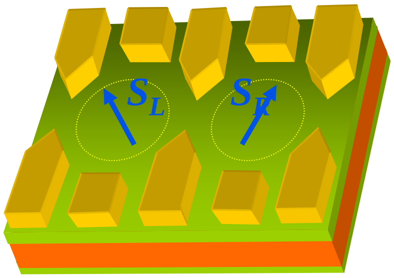
$$\mathcal{H}_J = \hbar \sum_{i<j} 2\pi J_{ij} \vec{I}^i \cdot \vec{I}^j = \hbar \sum_{i<j} 2\pi J_{ij} (I_x^i I_x^j + I_y^i I_y^j + I_z^i I_z^j)$$

$$\mathcal{H}_D = \sum_{i<j} \frac{\mu_0 \gamma_i \gamma_j \hbar}{4\pi |\vec{r}_{ij}|^3} \left[\vec{I}^i \cdot \vec{I}^j - \frac{3}{|\vec{r}_{ij}|^2} (\vec{I}^i \cdot \vec{r}_{ij})(\vec{I}^j \cdot \vec{r}_{ij}) \right]$$

Electron spin qubits

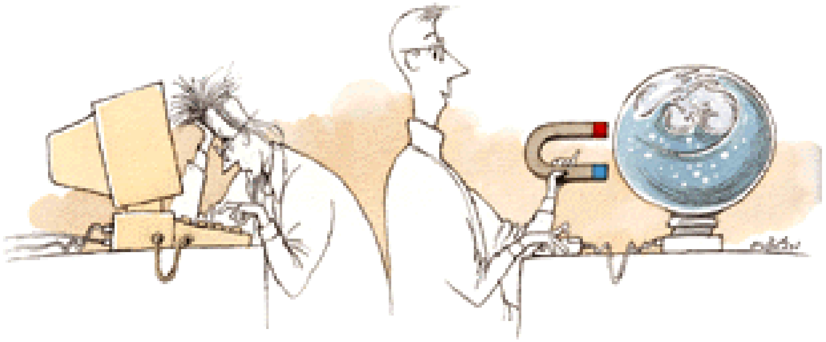


Kane, Nature 1998



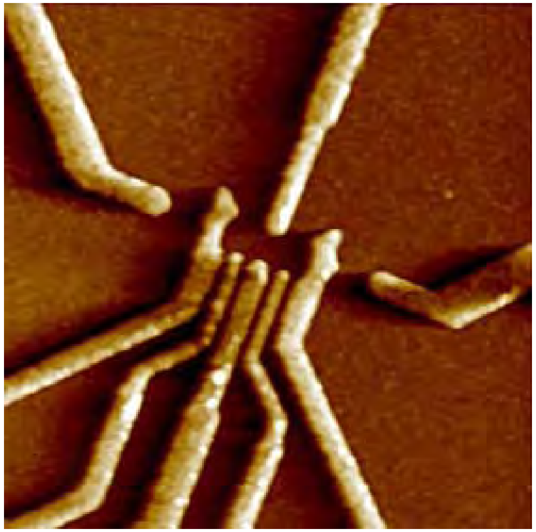
Loss & DiVincenzo, PRA 1998

References



Factoring 15	Vandersypen et al., *Nature* 414, 883 (2001)
Qubit control	Vandersypen & Chuang, *RMP,* to appear Oct 2004
Intro	Gershenfeld and Chuang, Scientific American, June 1998
Survey	Cory, et al., Fortschr. Phys. 48, 875, 2000.
Survey	Jones, Fortschr. Phys., 48, 909, 2000.
Survey	Cory, Chuang et al., recent preprint (arXiv.org/???)

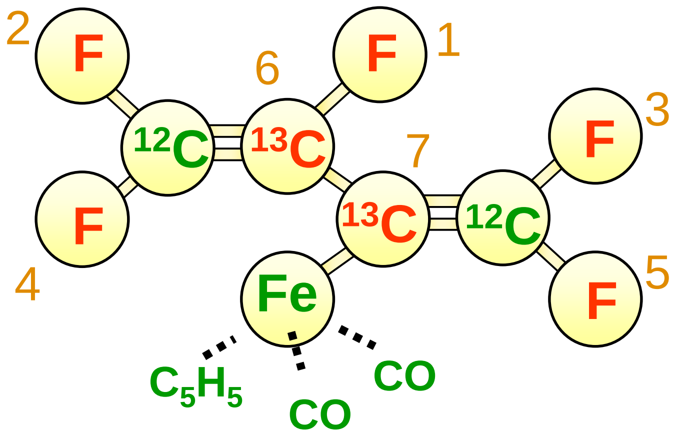
Quantum Information Processing with Semiconductor Quantum Dots



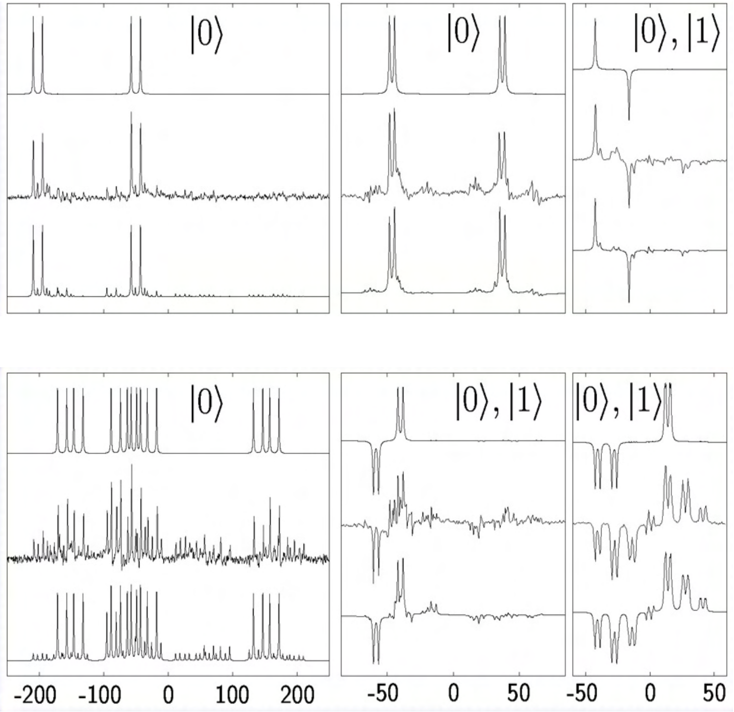
slides courtesy of Lieven Vandersypen, TU Delft

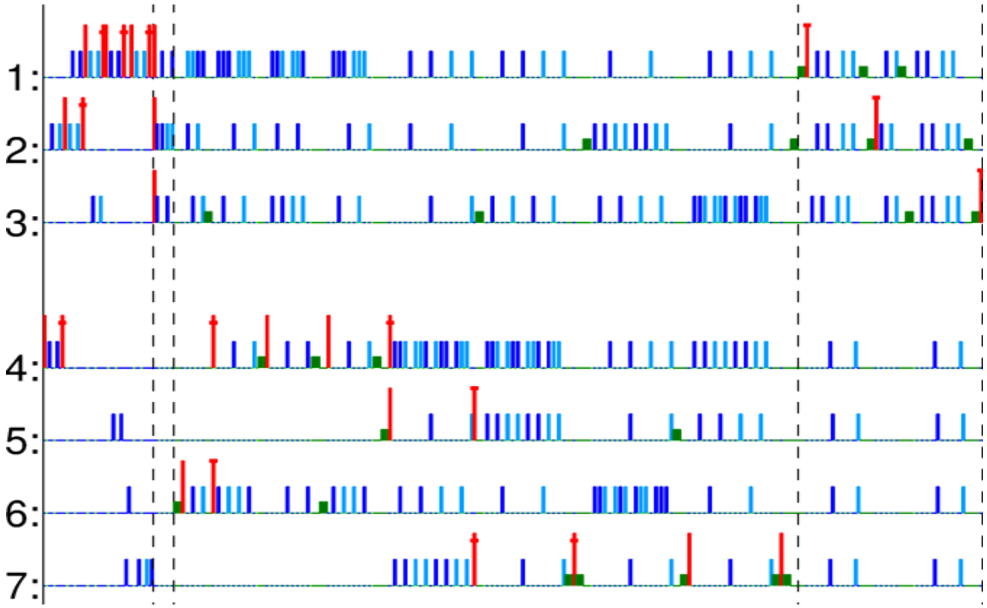
A seven-spin NMR quantum computer

Vandersypen et al., *Nature* **414**, 883 (2001)
Vandersypen & Chuang, *RMP*, Oct 2004.

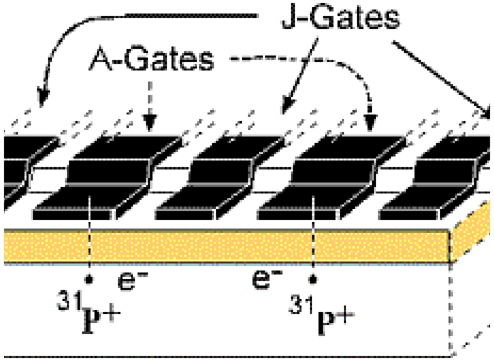


15 = 3 x 5

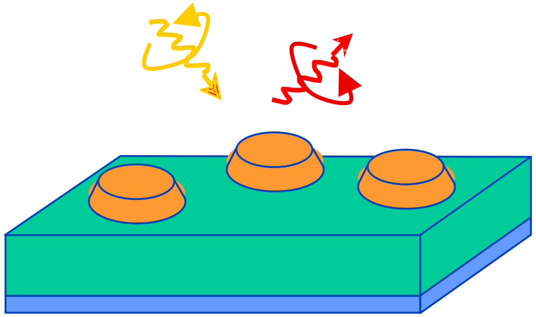




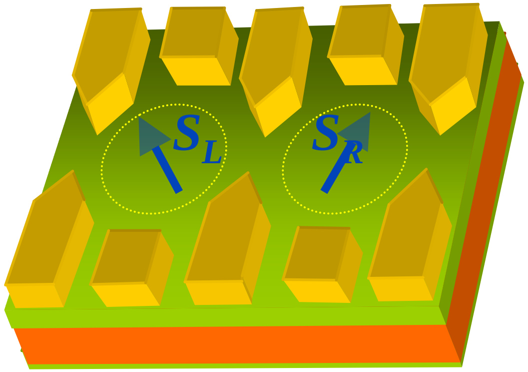
Can we access the quantum world at the level of single-particles? in a solid state environment?



Kane, Nature 1998



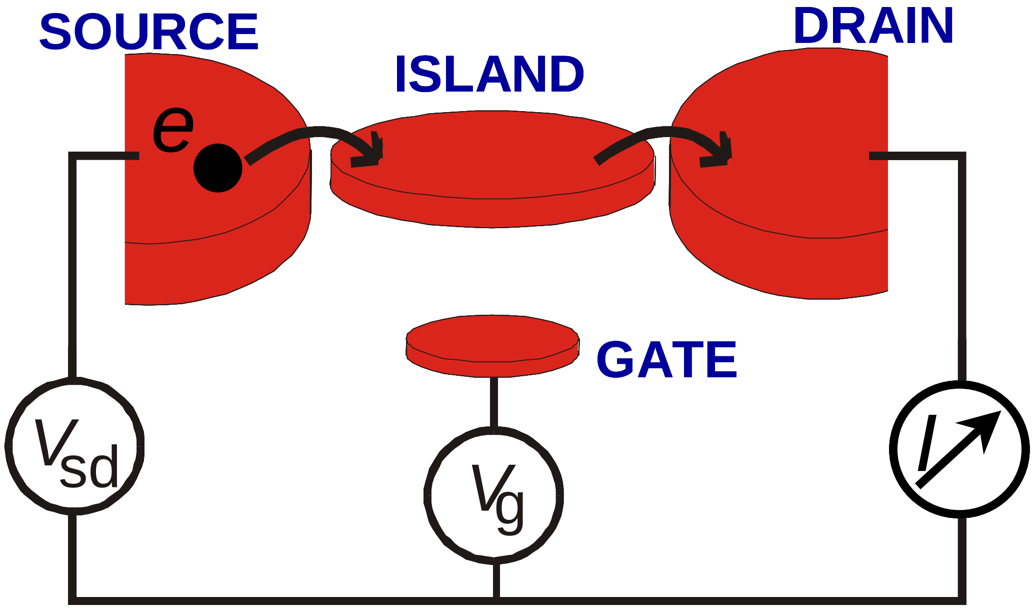
Imamoglu *et al*, PRL 1999



Loss & DiVincenzo
PRA 1998

Electrically controlled and measured quantum dots

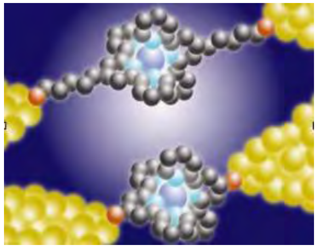
A small semiconducting (or metallic) island
where electrons are confined, giving a discrete level spectrum



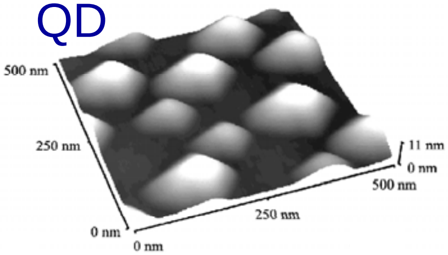
- Coupled via tunnel barriers to source and drain reservoirs
- Coupled capacitively to gate electrode, to control # of electrons

Examples of quantum dots

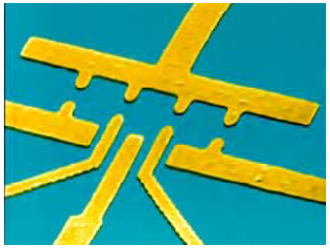
single molecule



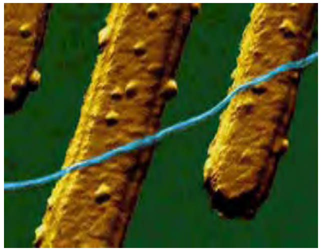
self-assembled QD



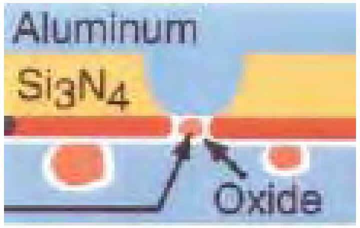
lateral QD



nanotube

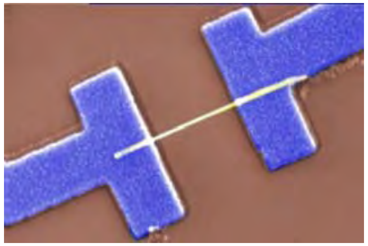


1 nm 10 nm 100 nm 1μm



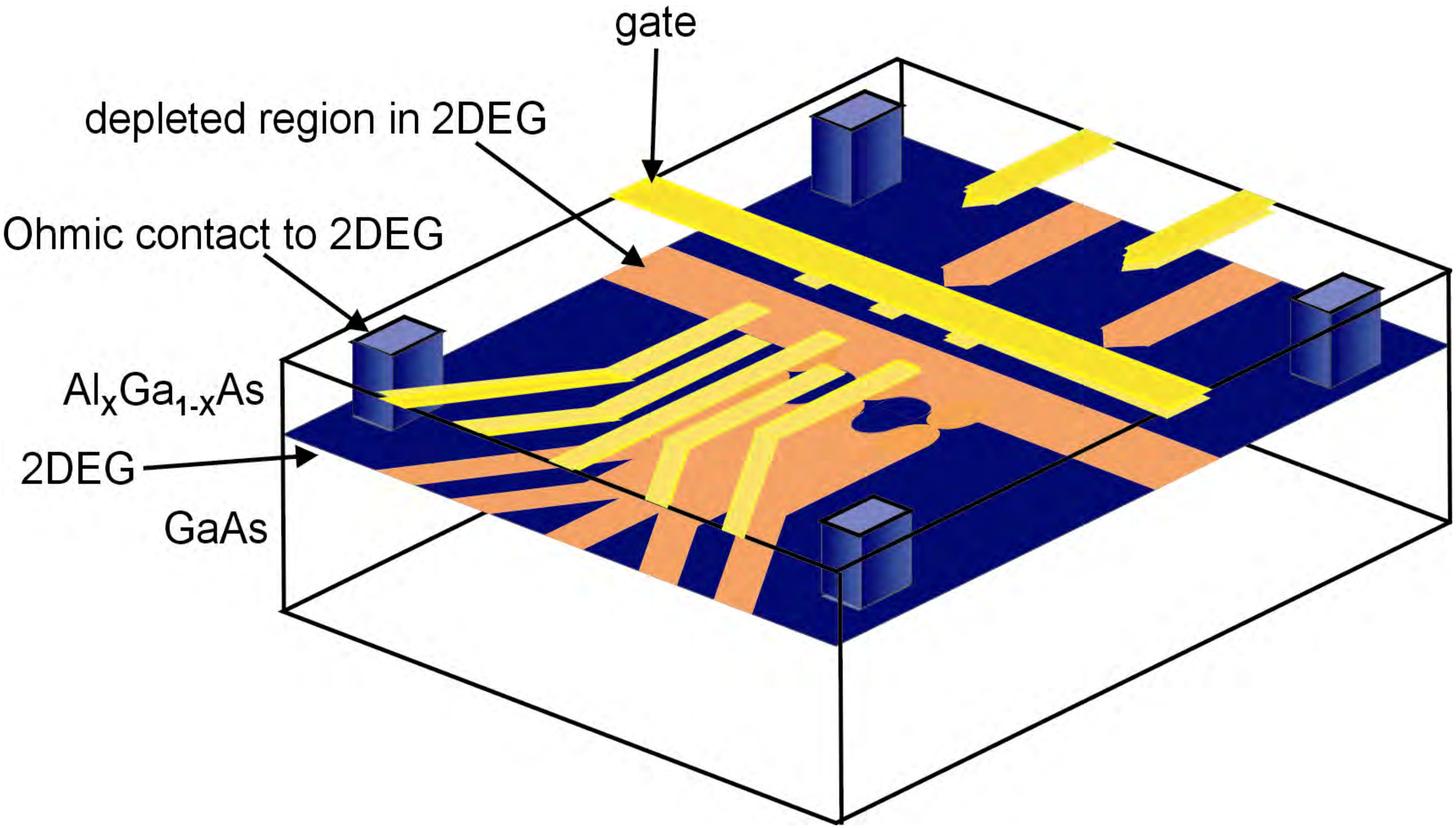
metallic nanoparticle

vertical QD



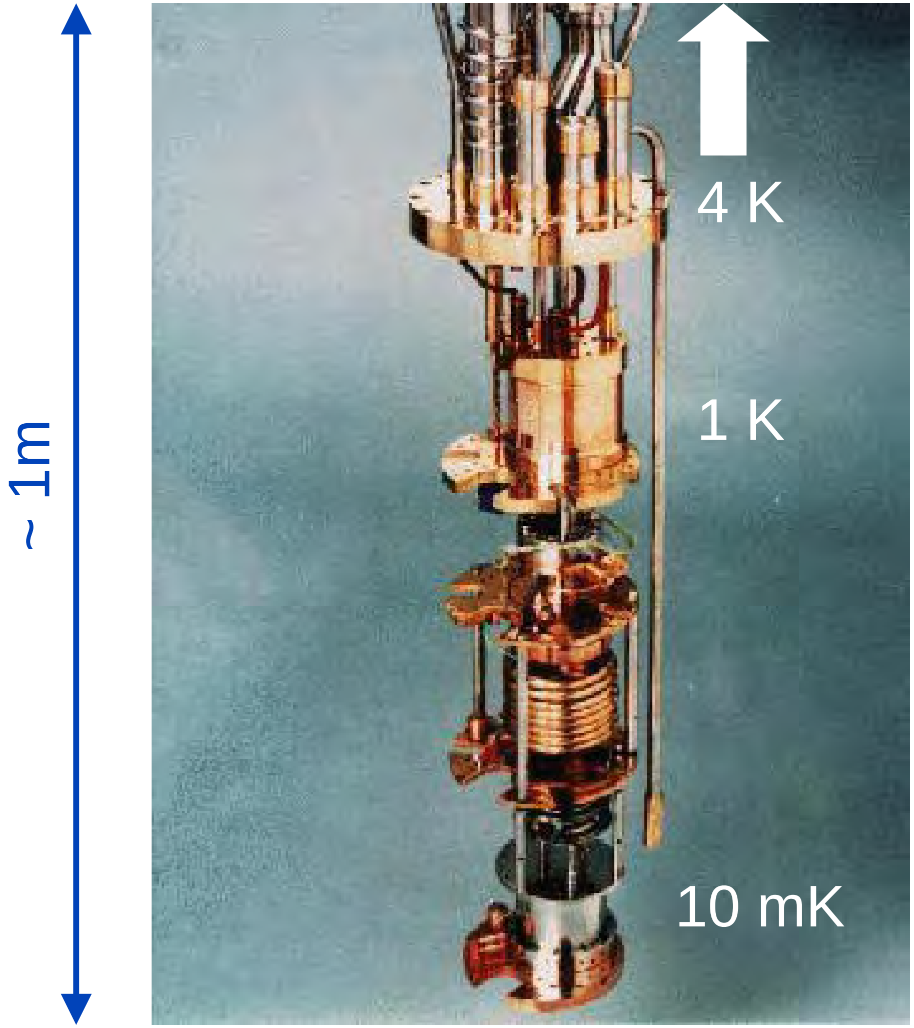
nanowire

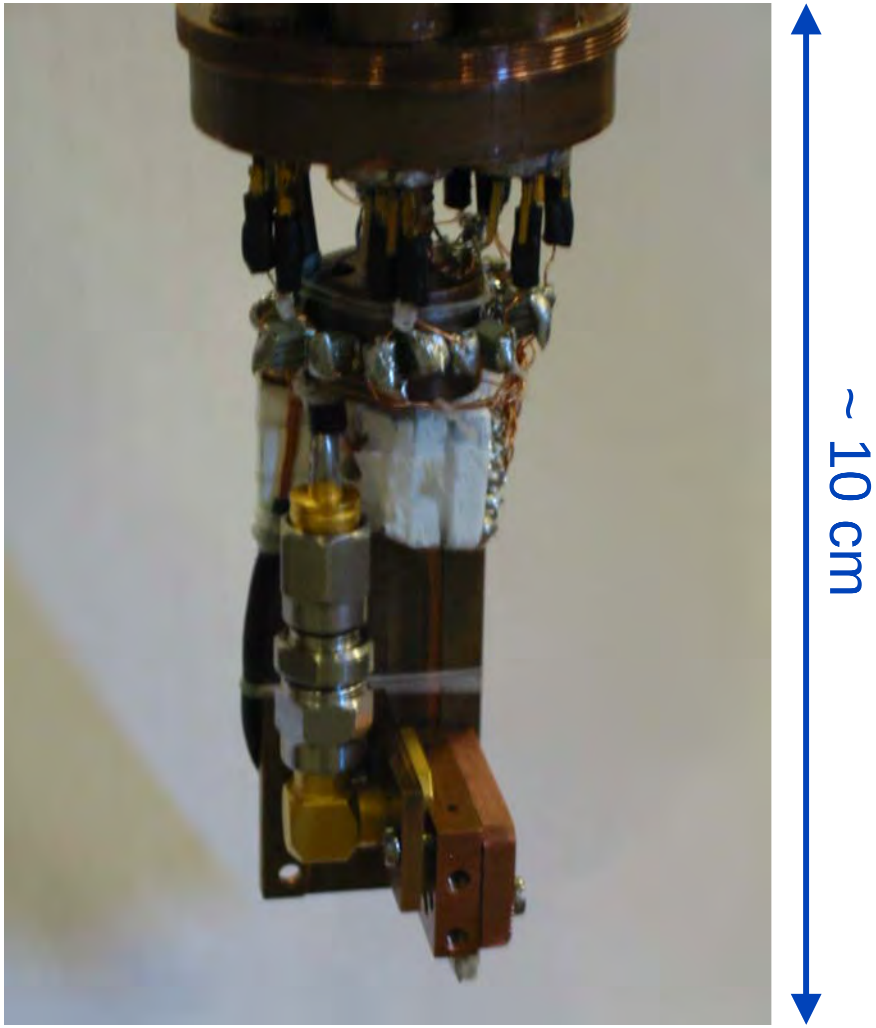
Electrostatically defined quantum dots



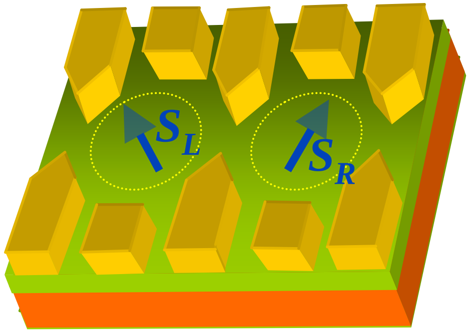
- Electrically measured (contact to 2DEG)
- Electrically controlled number of electrons
- Electrically controlled tunnel barriers

Dilution refrigerator





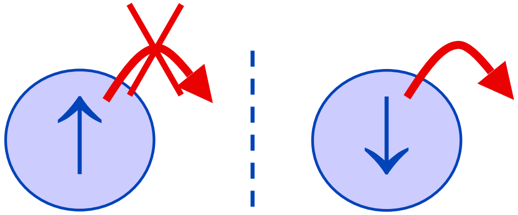
Spin qubits in quantum dots



Loss & DiVincenzo, PRA 1998
Vandersypen et al., Proc. MQC02 (quant-ph/0207059)

Initialization 1-electron, low T, high B_0

$$H_0 \sim \Sigma\ \omega_i\ \sigma_{zi}$$

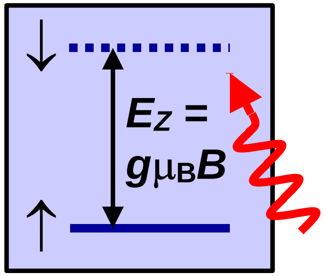


Read-out convert spin to charge

then measure charge

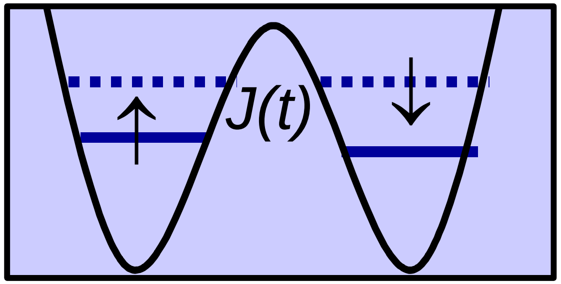
ESR pulsed microwave magnetic field

$$H_{RF} \sim \Sigma\ A_i(t)\ \cos(\omega_i\, t)\ \ \sigma_{xi}$$



SWAP exchange interaction

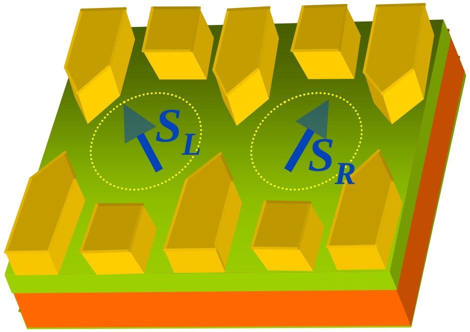
$$H_J \sim \Sigma\ J_{ij}\,(t)\ \ \sigma_i \cdot \sigma_j$$



Coherence long relaxation time T_1
long coherence time T_2

Spin qubits in quantum dots

Loss & DiVincenzo, PRA 1998
Vandersypen et al., Proc. MQC02 (quant-ph/0207059)

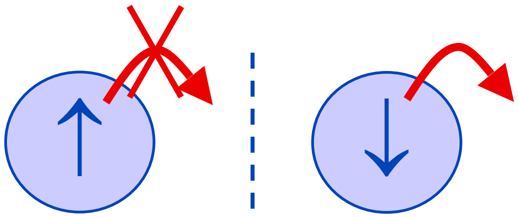


Initialization 1-electron, low T, high B_0

$$H_0 \sim \Sigma\, \omega_i\, \sigma_{zi}$$

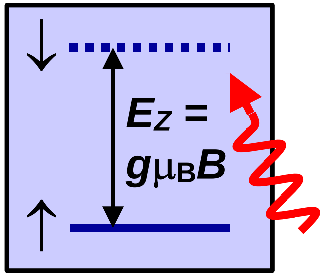
Read-out convert spin to charge

then measure charge



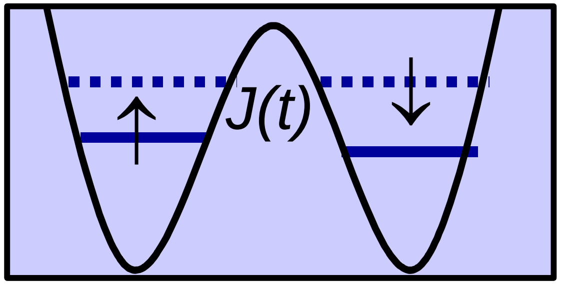
ESR pulsed microwave magnetic field

$$H_{RF} \sim \Sigma\, A_i(t) \cos(\omega_i t)\ \ \sigma_{xi}$$



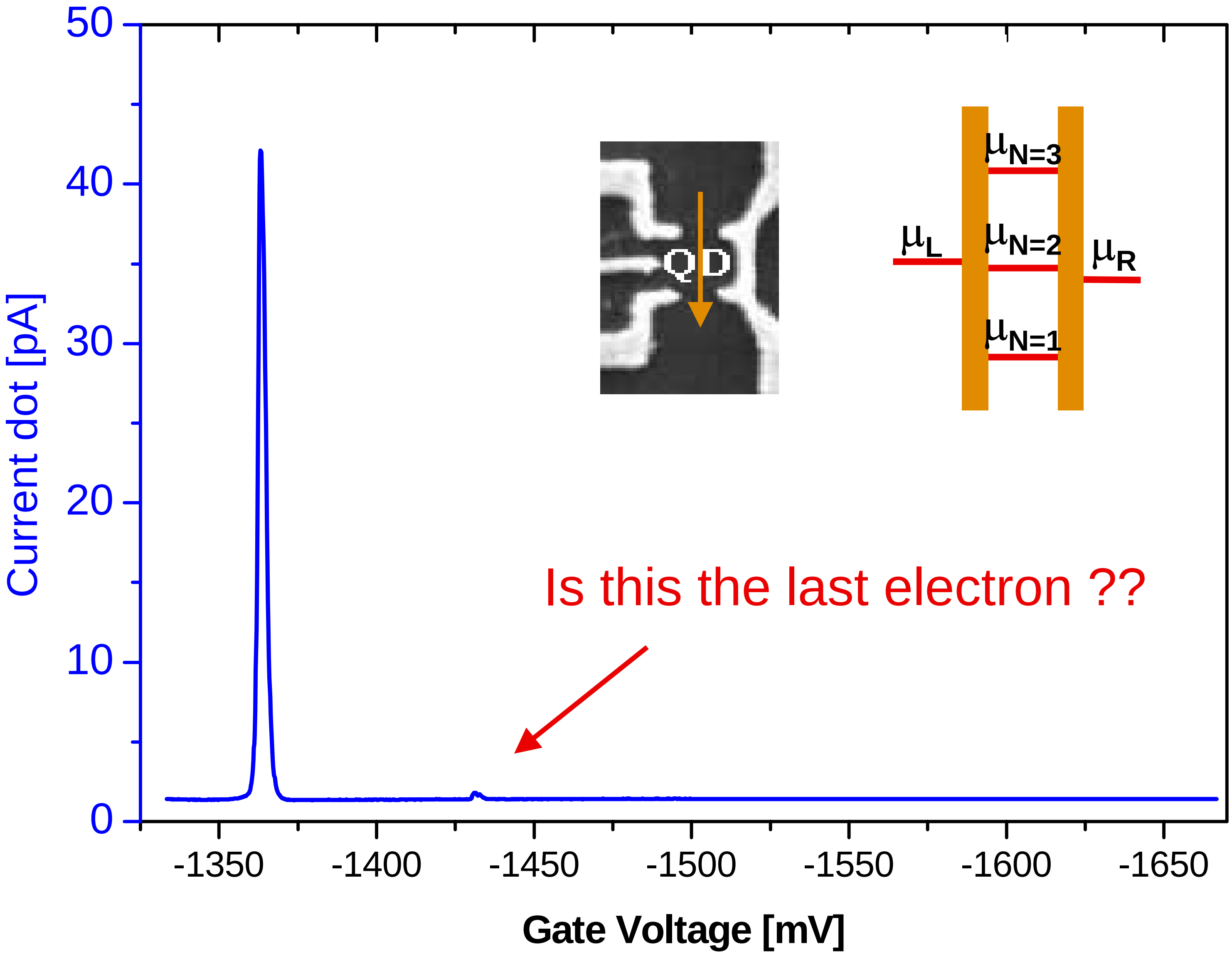
SWAP exchange interaction

$$H_J \sim \Sigma\, J_{ij}(t)\ \ \sigma_i \cdot \sigma_j$$



Coherence long relaxation time T_1
long coherence time T_2

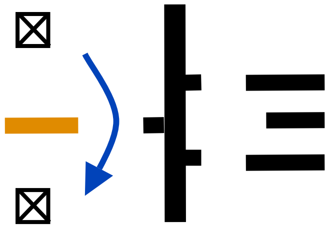
Transport through quantum dot - Coulomb blockade

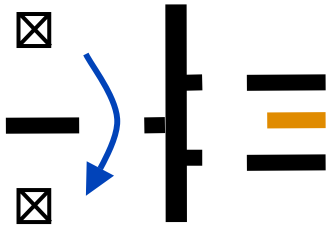


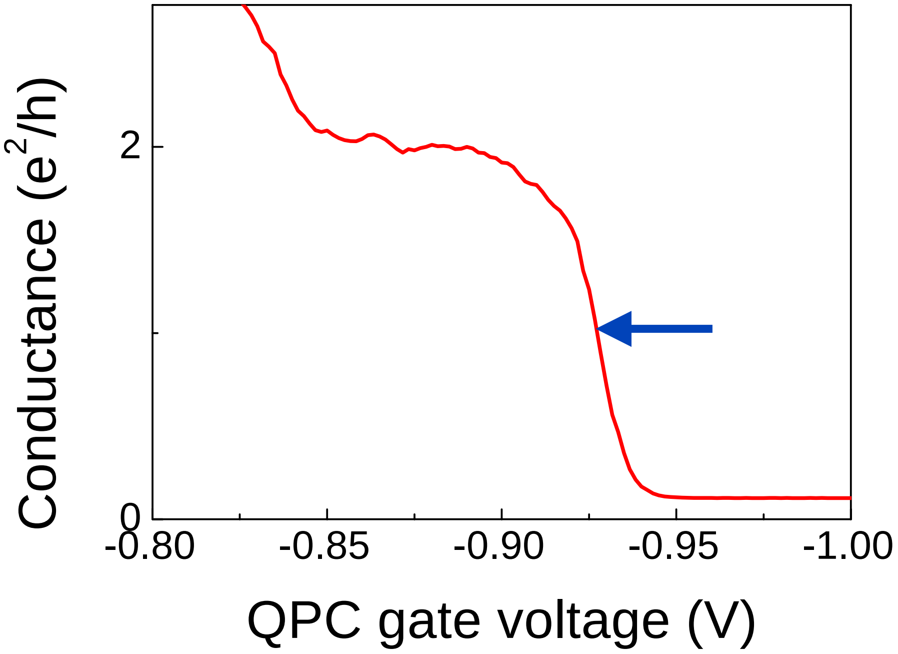
Is this the last electron ??

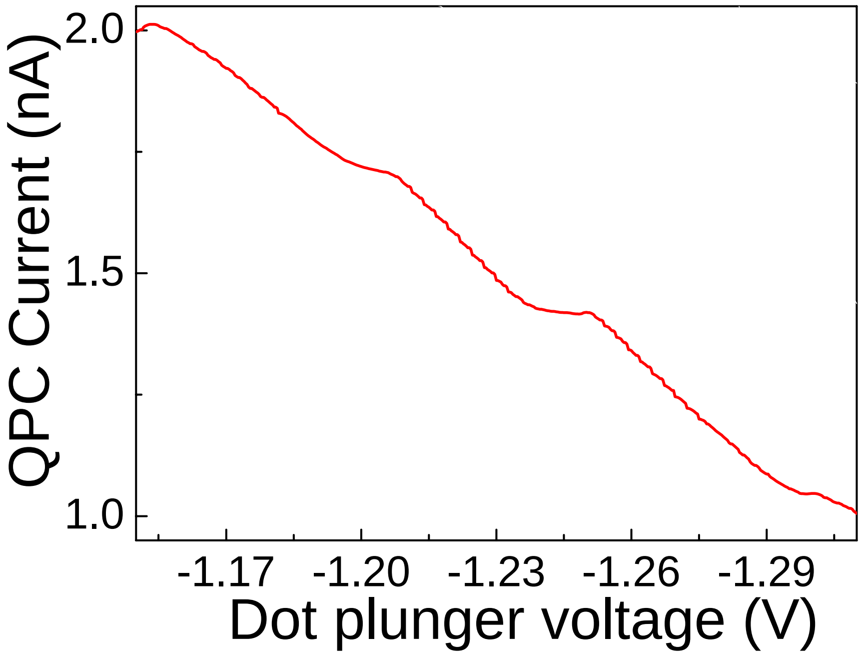
A quantum point contact (QPC) as a charge detector

Field *et al*, PRL 1993

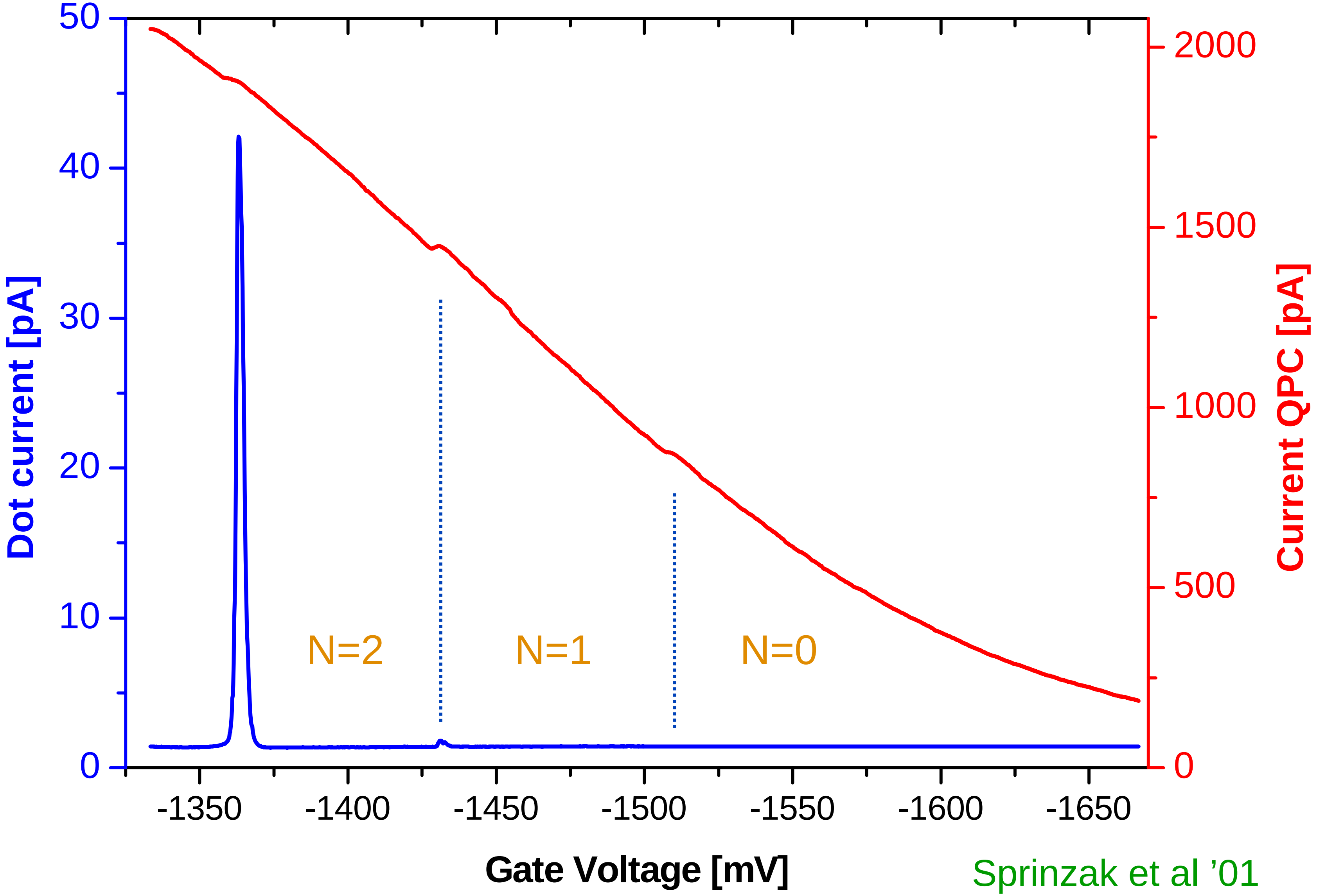








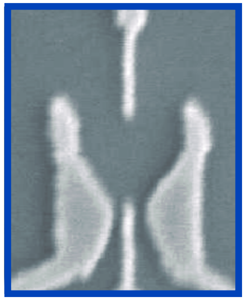
The last electron!



Sprinzak et al ’01

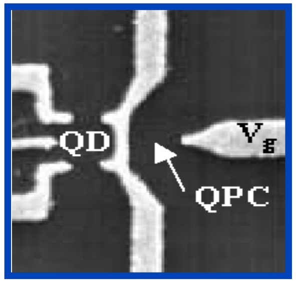
Few-electron double dot design

Ciorga et al '99



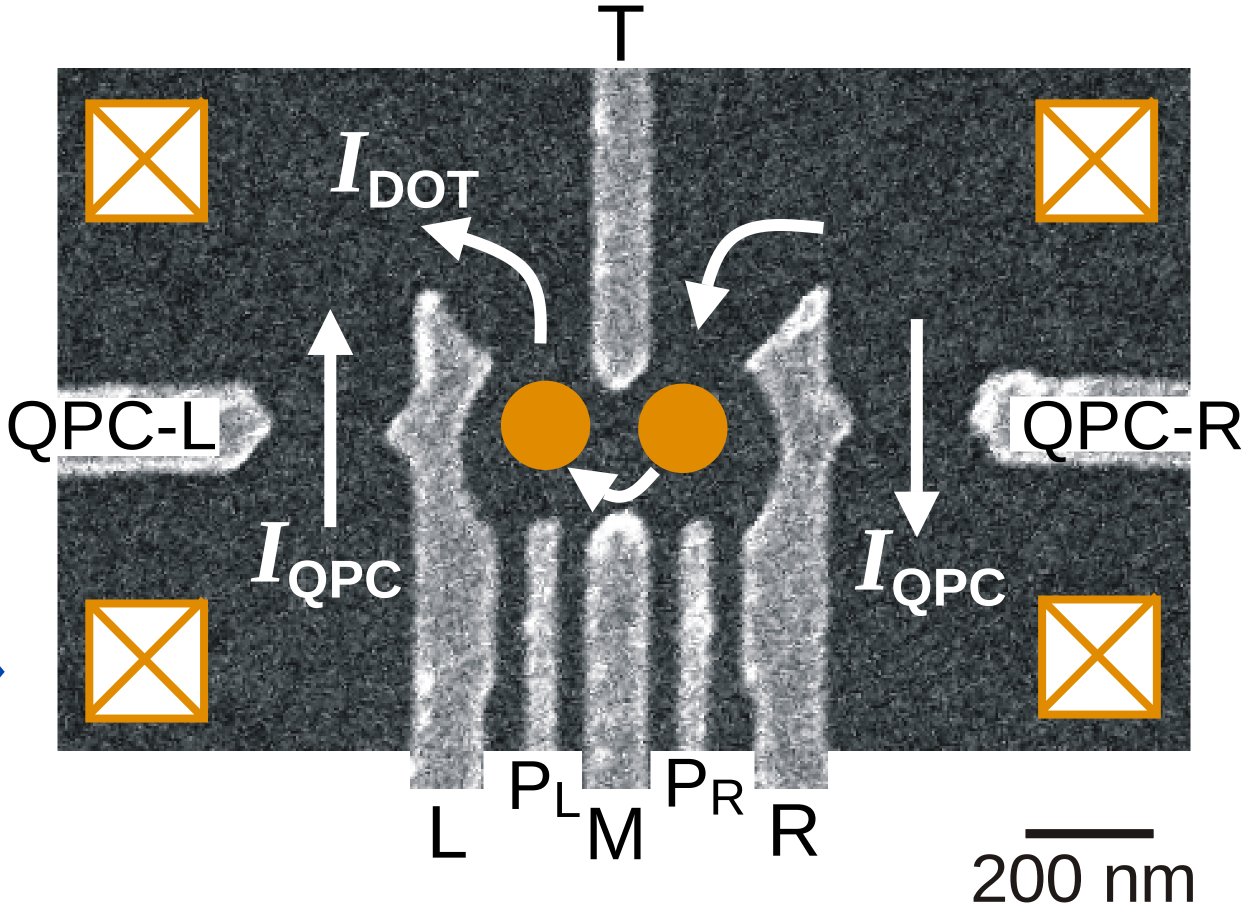
Open design

Field et al'93
Sprinzak et al '01



QPC for charge detection

Elzerman et al., PRB 2003

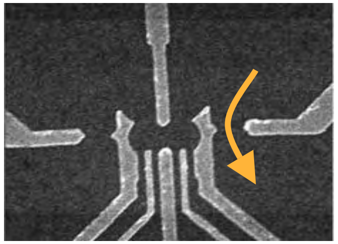


GaAs/AlGaAs wafers:

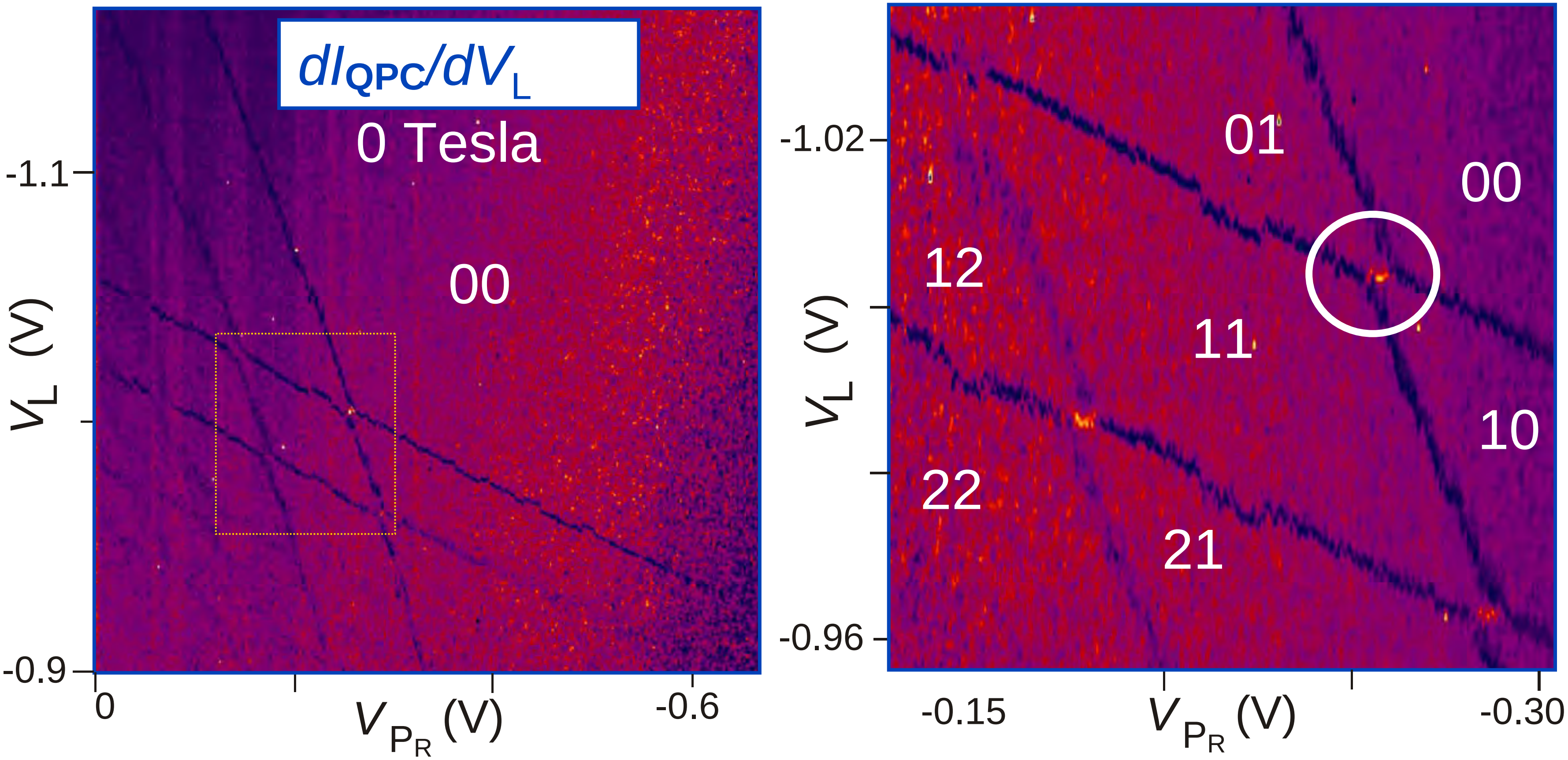
- NTT (T. Saku, Y. Hirayama)
- Sumitomo Electric
- Universität Regensburg (W. Wegscheider)

Few-electron double dot

Measured via QPC

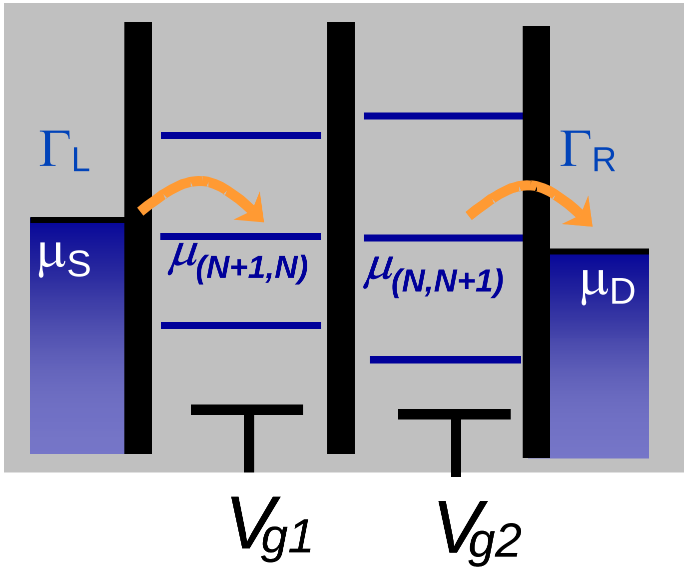


J.M. Elzerman et al., PRB **67**, R161308 (2003)



- Double dot can be emptied
- QPC can detect all charge transitions

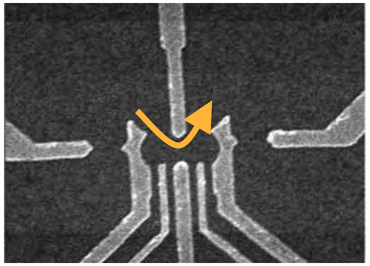
Single electron tunneling through two dots in series

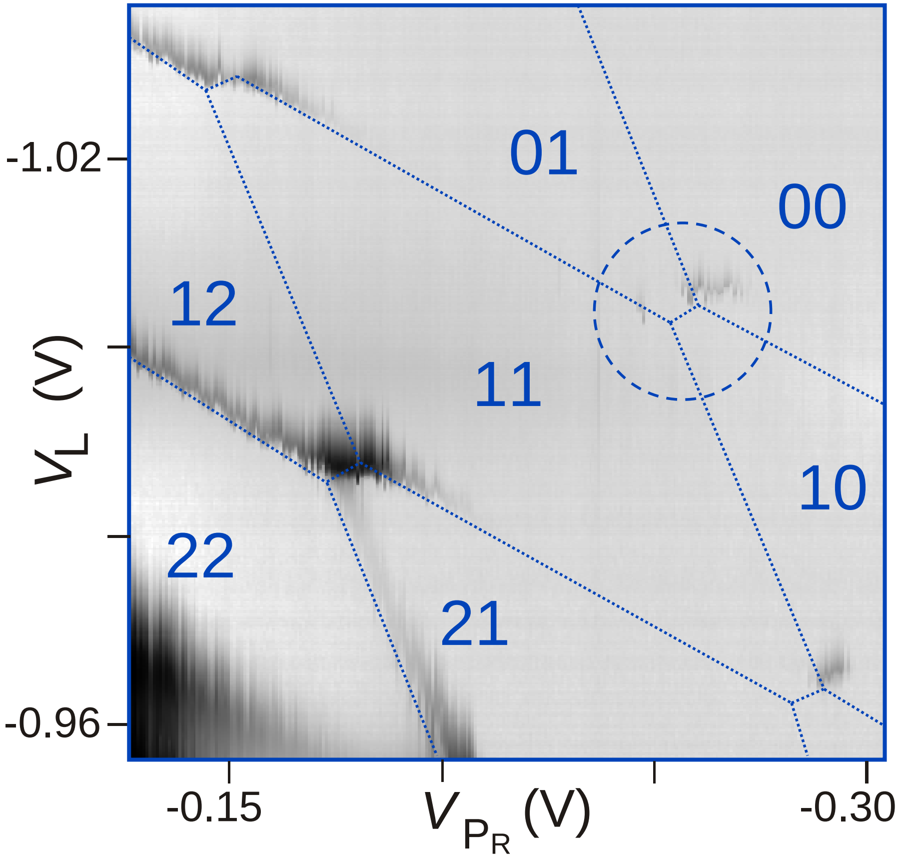


Few-electron double dot

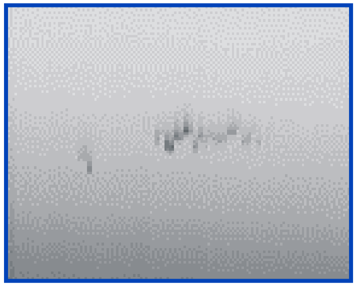
Transport through dots

J. Elzerman et al., cond-mat/0212489



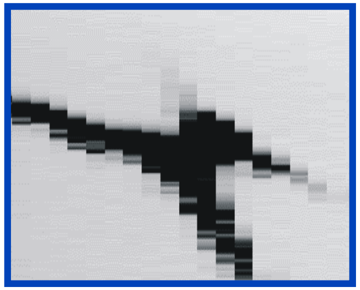


Peak height



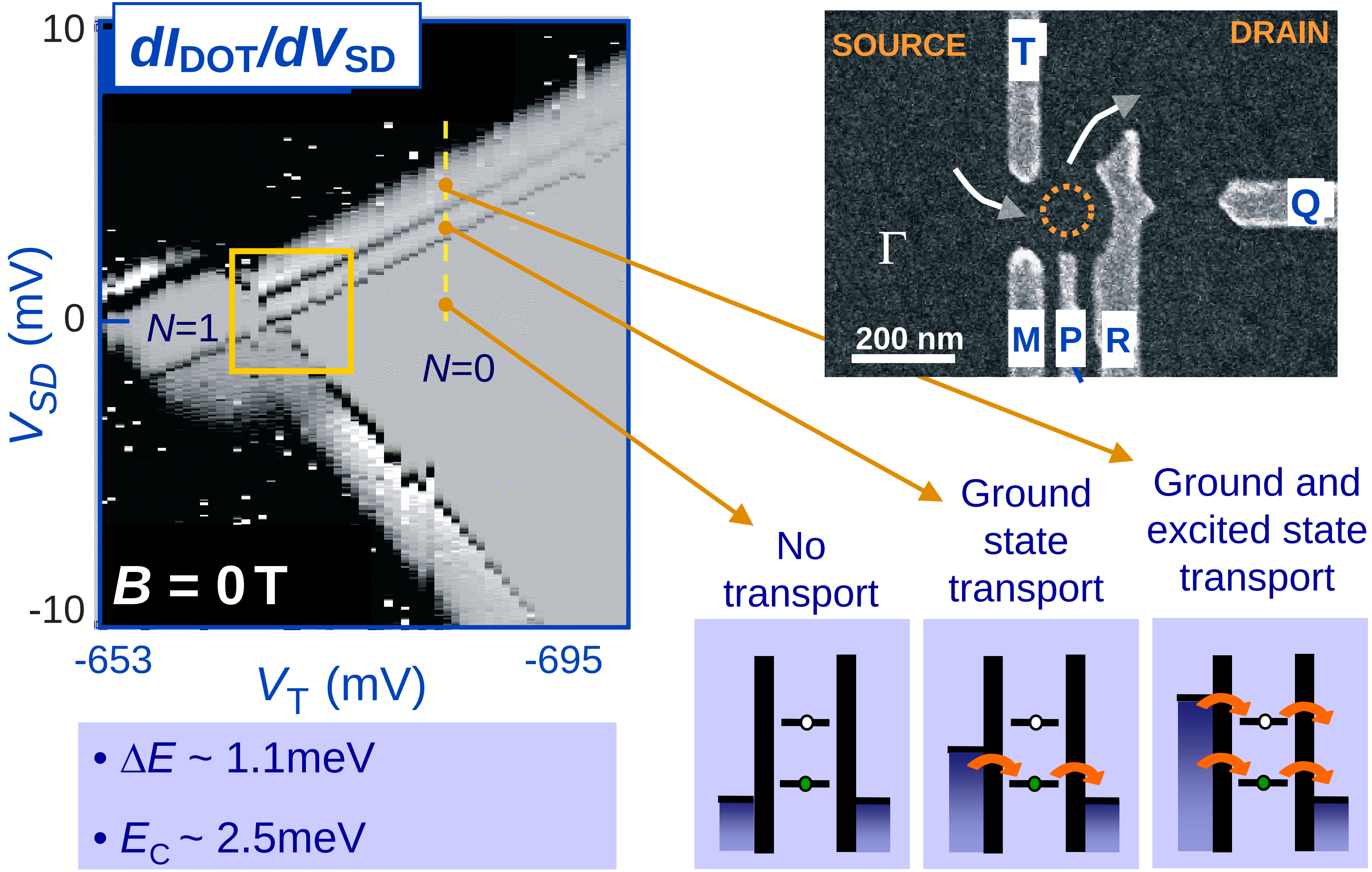
< 1 pA

2 pA



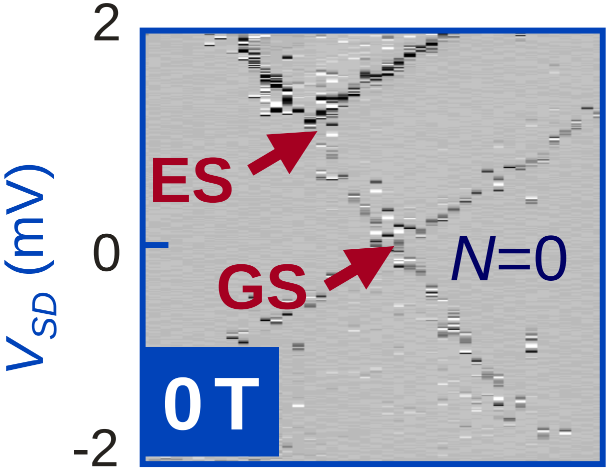
70 pA

Energy level spectroscopy at B = 0

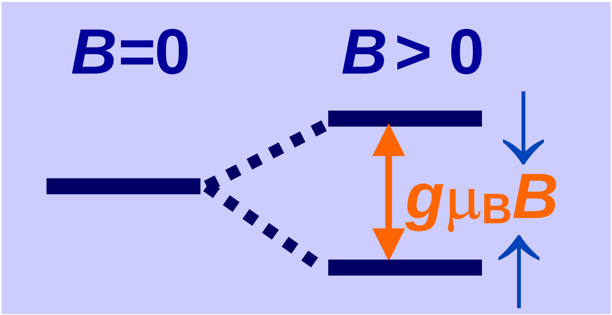


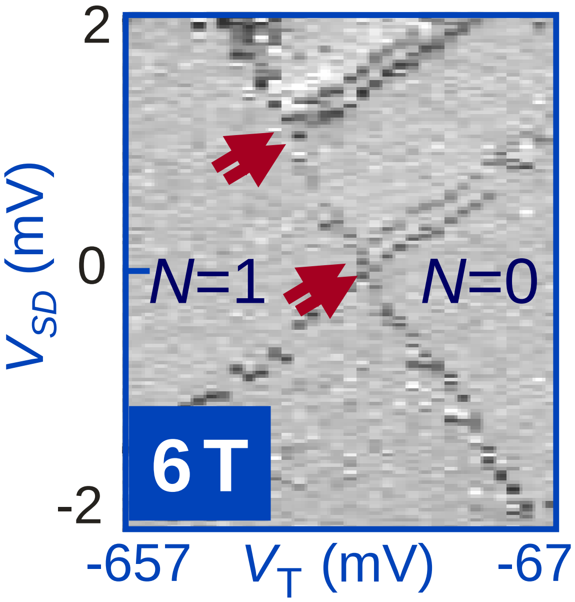
- ΔE ~ 1.1meV
- E_C ~ 2.5meV

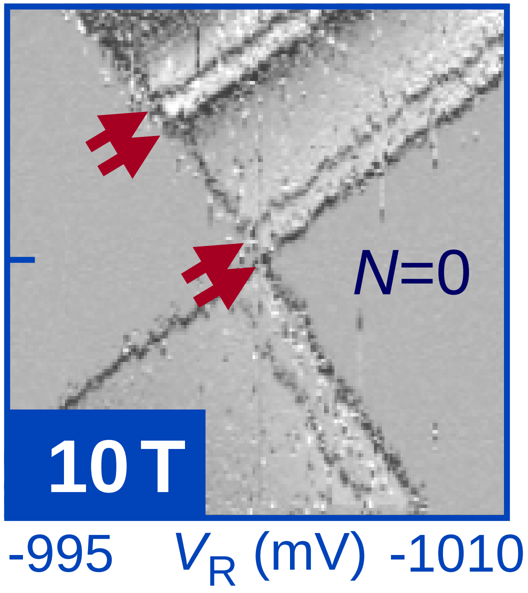
Single electron Zeeman splitting in $B_{//}$

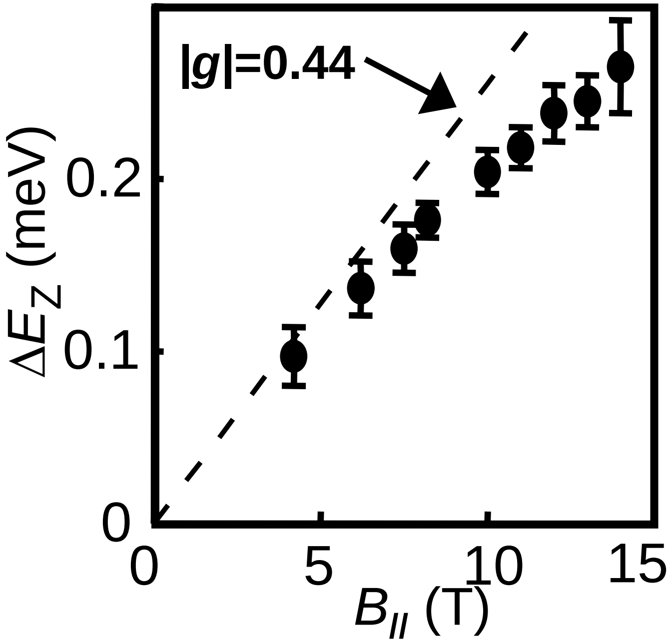


Hanson et al, PRL 91, 196802 (2003)
Also: Potok et al, PRL 91, 016802 (2003)



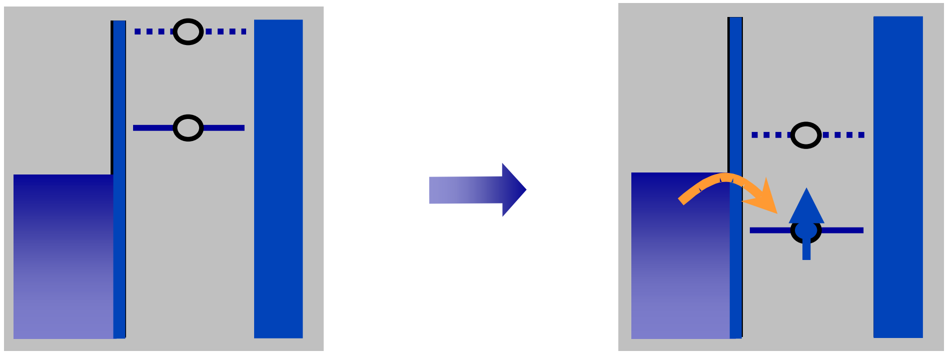




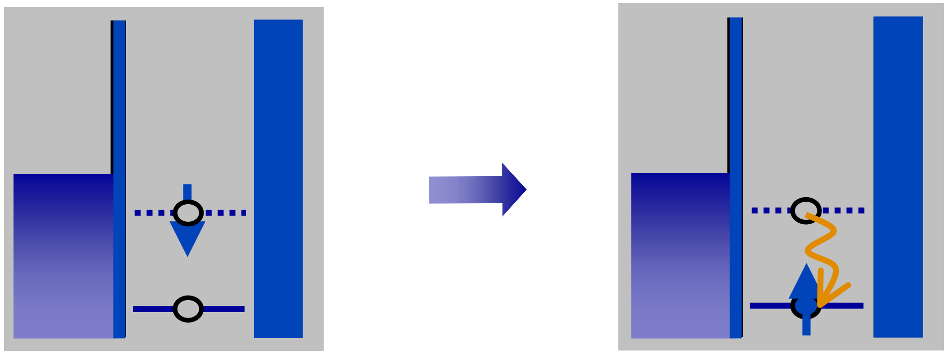


Initialization of a single electron spin

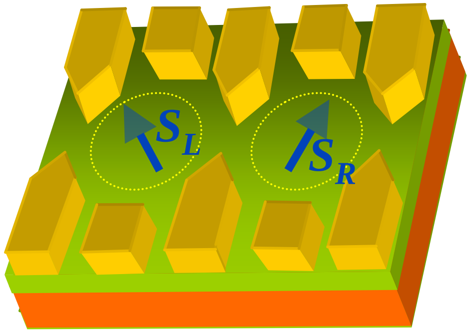
Method 1:
spin-selective
tunneling



Method 2:
relaxation to
ground state



Spin qubits in quantum dots



Loss & DiVincenzo, PRA 1998
Vandersypen et al., Proc. MQC02 (quant-ph/0207059)

Initialization	1-electron, low T, high B_0 $H_0 \sim \Sigma\ \omega_i\ \sigma_{zi}$
Read-out	convert spin to charge then measure charge
ESR	pulsed microwave magnetic field $H_{RF} \sim \Sigma\ A_i(t)\ \cos(\omega_i t)\ \ \sigma_{xi}$
SWAP	exchange interaction $H_J \sim \Sigma\ J_{ij}(t)\ \ \sigma_i \cdot \sigma_j$
Coherence	long relaxation time T_1 long coherence time T_2